Mémoire d'habilitation à diriger des recherches

Periodic groups and related topics

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Contents

Introduction

The notion of group is central in mathematics. It is the simplest algebraic structure used to describe the symmetries of a given object, such as the set of solutions of an equation, a geometric tiling, an algebraic structure, etc. Therefore it appears in many branches of mathematics. Over-simplifying modern research, there are two common strategies to explore the world of groups.

— One can start from a fixed group (e.g. Thomson group, $\mathrm{Out}(\mathbf{F}_r)$, etc) or class of groups (e.g. Fuchsian groups) and investigate all its properties.

— Conversely, one can choose a group property (such as amenability, Kazhdan Property (T), etc) and review "all" the groups in light of this property. What groups have (or not) the property? What are its connections to other properties?

The second approach is often a source of numerous exotic objects: if it is known that Property '*foo*' implies Property '*bar*', it is natural to look for a group that could satisfy '*bar*' but not '*foo*'. This leads to various "monster groups" (Golod-Schafarevich groups, Grigorchuk group, Tarski monsters, Gromov monster, etc) which are very nicely depicted as lions in Bridson's universe, see Figure 1. Those creatures are very valuable to check conjectures or test the robustness of the methods we develop. They are the common thread of our research, with a special inclination for torsion groups.

Periodic groups. A group G is *periodic* with *exponent* n (or n-*periodic*), if $g^n = 1$, for every $g \in G$. If G is finite, then by Lagrange's theorem, it is periodic. In 1902, Burnside asked whether the converse holds true [55]. More precisely, is every finitely generated, periodic group necessarily finite? To tackle this question, the first group to consider is the *free Burnside group* of rank r and exponent n defined by the presentation

$$\mathbf{B}_r(n) = \langle a_1, \dots, a_r \mid x^n,\ \forall x \rangle .$$

It is indeed the "largest" group of rank r and exponent n. For a long time, it was only known that $\mathbf{B}_r(n)$ is finite for some small exponents ($n = 2$ [55], $n = 3$ [55, 104], $n = 4$ [140] and $n = 6$ [89]). A major breakthrough in the subject was achieved by Novikov and Adian in 1968. They proved that $\mathbf{B}_r(n)$ is infinite provided $r \geqslant 2$ and n is a sufficiently large odd exponent [116]. Later Ol'shanskiĭ provided an alternative proof of the same result [117]. Despite these progresses the case of *even* exponents held up longer. It was only in the early nineties that Ivanov [94] and Lysenok [109] independently proved that free Burnside groups of sufficiently large even exponents are also infinite.

The class of n-periodic groups forms a variety of groups, in the sense of Neumann [114], which we denote by $\mathfrak{B}_n$. Its free elements are the free Burnside groups (of exponent n). The study of periodic groups (and other related monsters) has been a motivation for many developments in geometric group theory. Nevertheless, unlike other varieties, such as abelian groups, nilpotent/solvable groups of a given class/length, etc, $\mathfrak{B}_n$ has received less attention. In our work we attempt to have a

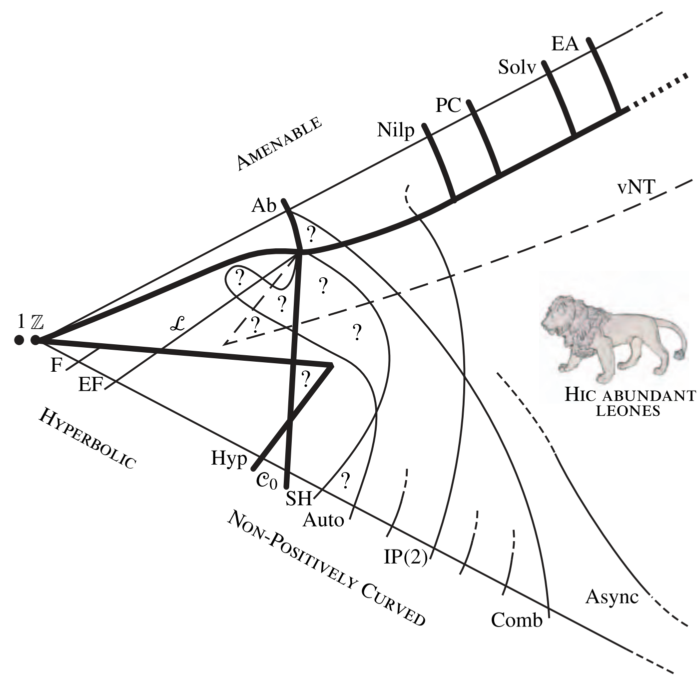


Figure 1 – The universe of groups according to Bridson [50]

systematic approach of $\mathfrak{B}_n$. The initial steps of this program, which are detailed in Chapter 1, are the following.

1. The first task was to get a deeper understanding why finitely generated periodic groups can be infinite. In [71], Delzant and Gromov provided an alternative proof of the Novikov-Adian theorem using a geometric approach of small cancellation theory. Nevertheless, their results only apply to groups with odd exponents. We revisited their work and adapted it to the case of even exponents [9]. We recovered along the way a result by Ol'shanskiĭ and Ivanov stating that every non-elementary hyperbolic group admits infinite periodic quotients [95]. The method actually generalizes to study (partially) periodic quotients of any group G which admits a reasonable action on a Gromov hyperbolic space [6, 9].

2. It is practically impossible to study a class of groups without having numerous test examples at our disposal. Producing a single new infinite, finitely generated group in $\mathfrak{B}_n$ can be cumbersome. Indeed, most of the time, it requires to go through all the steps needed to prove that infinite periodic groups exist. With Dominik Gruber, we developed a small cancellation theory in the variety $\mathfrak{B}_n$ [13]. It provides a versatile yet powerful tool that can be used without prior knowledge in the field. As an example, we apply it to produce a periodic version of the Gromov monster, that is a finitely generated periodic group whose Cayley graph coarsely contains an expander graph, and therefore does not coarsely embed in any Hilbert space. In particular, this group does not satisfy the Baum-Connes conjecture with coefficients or Yu's property (A).

3. In 1911, Dehn suggested three decision problems (the word, conjugacy, and isomorphism problems) injecting mathematical logic into group theory [70]. Novikov and Boone indepen-

dently produced examples of finitely presented groups with unsolvable word problem [115, 45]. Starting from there, Adian and Rabin showed that most decision problems are unsolvable in general [26, 135]. However they often admit a solution, if we restrict ourselves to a specific class of groups. This motivated us to study decision problems in the Burnside variety. If n is a sufficiently large integer which is not prime, Kharlampovich proved that there exists a finitely generated group in $\mathfrak{B}_n$ for which the word problem is not solvable. Building on this example, we proved a version of the Adian-Rabin theorem in $\mathfrak{B}_n$ [13]. It states that given any group property stable under taking subgroups, there is no algorithm that can decide whether a group $G \in \mathfrak{B}_n$ has this property or not. From an algorithmic point of view, it means that $\mathfrak{B}_n$ is not easier to handle that the class of all groups.

In addition to the aforementioned results, we investigated other aspects of periodic groups (automorphisms of periodic groups, algebraic constructions preserving $\mathfrak{B}_n$, etc). During this journey, we encountered various topics of geometric group theory. Two of them, connected to dynamical systems, particularly caught our attention, namely growth of automorphisms and growth of groups. So, we decided to explore them in a larger context.

Growth of automorphisms. If we want to fully capture the essence of free Burnside groups, we need to understand their symmetries. This lead us to the study of outer automorphisms of periodic groups. Any automorphism ϕ of the free group $\mathbf{F}_r$ sends an n-th power to an n-th power. It follows that the projection $\mathbf{F}_r \twoheadrightarrow \mathbf{B}_r(n)$ induces a homomorphism $\chi\colon \operatorname{Out}(\mathbf{F}_r) \to \operatorname{Out}(\mathbf{B}_r(n))$. This map is a source of numerous examples of outer automorphisms of $\mathbf{B}_r(n)$ [4]. However it is neither one-to-one nor onto, which raises the problem of describing its kernel and image. For the former one, we adopted an asymptotic point of view and asked the following question: what automorphism of $\mathbf{F}_r$ induces an infinite order automorphism of $\mathbf{B}_r(n)$, provided n is sufficiently large? It turns out that the solution builds an unexpected connection with growth of automorphisms.

Let G be a finitely generated group endowed with the word metric. For any conjugacy class c of G, denote by $\|c\|$ the length of the smallest element in c. The outer automorphism group of G naturally acts on the set of conjugacy classes of G. Assume that $G = \mathbf{F}_r$ is a free group. Given an automorphism $\Phi \in \operatorname{Out}(G)$ and a conjugacy class c of G, one observes a *growth dichotomy*: the map $k \mapsto \|\Phi^k(c)\|$ grows either polynomially or at least exponentially.

With Arnaud Hilion, we proved that for any automorphism $\Phi \in \operatorname{Out}(\mathbf{F}_r)$ the following are equivalent [16].

1. There exists a conjugacy class c of $\mathbf{F}_r$ such that the map $k \mapsto \|\Phi^k(c)\|$ grows at least exponentially.
2. There exists $N \in \mathbf{N}$ such that for every odd exponent $n \geqslant N$, the automorphism of $\mathbf{B}_r(n)$ induced by Φ has infinite order.

This result motivated us to explore further outer automorphism groups. In particular, we wondered, what are the groups whose automorphisms satisfy the same growth dichotomy. We already mentioned free groups. Other known examples are free abelian groups and surface groups. In view of these examples, the next groups to look at are hyperbolic groups and toral relatively hyperbolic groups. With Camille Horbez, Gilbert Levitt and Arnaud Hilion we proved that their automorphisms satisfy the same growth dichotomy [17]. In the opposite direction, we provided examples of finitely generated groups G and automorphisms $\Phi \in \operatorname{Out}(G)$ for which the map $k \mapsto \|\Phi^k(c)\|$ can have almost any possible asymptotic behavior [10]. These ideas are detailed in Chapter 2.

Growth spectrum of groups. As previously, our investigation of growth started with periodic groups. Assume that G is a non-elementary, torsion-free hyperbolic group. Given an integer n we denote by G^n the (normal) subgroup of G generated by the n-th power of all its elements. If n is a sufficiently large odd exponent, Ol'shanskiĭ proved that G/G^n is infinite [118]. The next question is how "large" these periodic quotients are. It can be measured by the exponential growth rate. If G acts properly on a metric space X, the *exponential growth rate* of this action is

$$h(G,X) = \limsup_{r\to\infty} \frac{1}{r}\log\left|\{g\in G \ : \ d\left(gx,x\right)\leqslant r\}\right|.$$

(We denote by $|U|$ the cardinality of a set U.) Note that if N is a normal subgroup of G, then

$$h(G/N, X/N) \leqslant h(G,X).$$

Hyperbolic groups are growth tight, that is if the action of G on X is proper and co-compact, then the previous inequality is strict whenever N is infinite. Nevertheless, we proved that $h(G/G^n, X/G^n)$ can be made arbitrarily close to $h(G,X)$ [3]. More precisely, if G is a torsion-free hyperbolic group acting properly co-compactly on X, then there exists $\kappa \in \mathbf{R}_+^*$ such that for every odd integer n, we have

$$h(G,X) - \frac{\kappa}{n} \leqslant h(G/G^n, X/G^n) \leqslant h(G,X).$$

A dual way to measure the "size" of G/G^n is by estimating how small the kernel of the projection $G \twoheadrightarrow G/G^n$ is. This was done by Adian for free Burnside groups [29]. His initial goal was to show that $\mathbf{B}_r(n)$ is non-amenable (provided that $r\geqslant 2$ and n is a sufficiently large odd integer). The argument relies on a variation of Kesten's amenability criterion established independently by Grigorchuk and Cohen [79, 59]. This criterion is the following. Assume that X is the Cayley graph of $\mathbf{F}_r$ with respect to a free basis. Then for every normal subgroup N of $\mathbf{F}_r$, the quotient G/N is amenable if and only if $h(G,X) = h(N,X)$. Adian proved by hand that $h(\mathbf{F}_r^n, X) < h(\mathbf{F}_r, X)$ therefore showing that $\mathbf{B}_r(n)$ is non-amenable.

Assume now that G is a non-elementary torsion-free hyperbolic group, and X is a Cayley graph of G. If n is sufficiently large, it is now known that G/G^n admits infinite quotients with Kazhdan Property (T). This implies that G/G^n is not amenable, providing an alternative proof of Adian's result. It raises the following question: do we still have $h(G^n,X) < h(G,X)$? More generally, does the Grigorchuk-Cohen amenability criterion extend to all hyperbolic groups?

These questions lead us to the more general study of growth in groups. Given a group G acting by isometries on metric space X, it is common to define two growth spectra. The *quotient spectrum* is the set of growth rates of all quotients of G, i.e.

$$\{h(G/N, X/N) \ : \ N \lhd G, \ \text{normal subgroup}\}.$$

The *subgroup spectrum* is the set of growth rates of all subgroups of G, that is

$$\{h(H,X) \ : \ H < G, \ \text{subgroup}\}.$$

We are interested in the content of those set, and their relations with the algebraic properties of G. We refer for instance to Grigorchuk and de la Harpe [80] for various problems in the area. In our work, we focused on the extremal values of these spectra.

One achievement in this direction is a far-reaching generalization of the Grigorchuk-Cohen amenability criterion. Consider a group G acting properly by isometries on a Gromov hyperbolic space X. We suppose that this action is *strongly positively recurrent.* This assumption is made to study actions which are not convex co-compact but, from a "statistical" point of view, not far from being so. Together with Rhiannon Dougall, Barbara Schapira and Samuel Tapie we proved that if H is a subgroup of G then H is co-amenable in G if and only if $h(G, X) = h(H, X)$. This statement is the last step of a long history of research that started simultaneously by Grigorchuk and Cohen (in the case of free groups) on one hand and Brooks (for fundamental groups of hyperbolic manifolds) on the other hand. Our statement is optimal, in the sense that no hypotheses can be removed without hitting a counter-example. Growth problems also have connections to various representation theoretic rigidity properties, e.g. Kazhdan Property (T). We develop these ideas in Chapter 3.

Perspectives. In Chapter 4 we review some research projects that caught our attention. To be consistent with our main theme, we selected the ones which are related to periodic groups.

From a geometric point of view, one current difficulty to study free Burnside groups (or other infinite periodic groups) is that we do not know a suitable metric space on which they act. One is often forced to approximate $\mathbf{B}_r(n)$ by hyperbolic groups, as it is done to show that $\mathbf{B}_r(n)$ is infinite. Typical questions are: can a free Burnside group act properly on a CAT(0) space? Or even on an (infinite dimensional) CAT(0) cube complex? Is there another form of negative curvature that would be adapted for studying $\mathbf{B}_r(n)$? Answering one of them would have important consequences for periodic groups. In particular, it would give a more direct grip on them.

Meanwhile, we would like to borrow tools from other fields and build connections between Burnside varieties and other branches of mathematics. For instance, model theory (do all non-abelian free Burnside groups of a given exponent have the same first order theory?), dynamical systems and ergodic theory (what is the behavior of random walks in $\mathbf{B}_r(n)$?), representation and operator theory (does $\mathbf{B}_r(n)$ have the Haagerup property? the rapid decay property?), etc. We are convinced that it will not only shed a new light on Burnside varieties, but also lead to new developments in geometric group theory.

Illustrating Mathematics. In the recent years we have been interested in another field of research: visualization of mathematics. The goal is to produce tools (images, videos, 3D printed objects, software, etc) that help us understand mathematical structures. They are intended for researchers as well as for a broader audience interested in modern mathematics. In Appendix A we report on one of these projects conducted with Sabetta Matsumoto, Henry Segerman and Steve Trettel.

Thurston's geometrization conjecture (proved by Perelman) states that every closed three-manifold may be cut into finitely many pieces, each of which can be built from some homogeneous geometry. There are eight possible geometric structures involved, corresponding to the Euclidean space $\mathbf{E}^3$, the three-sphere S^3, the hyperbolic space $\mathbf{H}^3$, the product geometries $S^2 \times \mathbf{E}$ and $\mathbf{H}^2 \times \mathbf{E}$, and the Lie groups Nil, Sol as well as the universal cover of $\mathrm{SL}(2, \mathbf{R})$. To gain more insight, we developed a web application that simulates what would be the perception of a person living in one of these geometries or a quotient of it. This work relies on a relatively new computer graphics technique called ray-marching. It allows us to produce both accurate and real-time animations. In particular, the virtual reality version allows a complete immersion in Thurston's geometries.

Chapter 1

Periodic groups

In this chapter, we review several techniques to produce infinite periodic groups, with a particular emphasis on the case of even exponent.

1.1 Geometric small cancellation theory

Small cancellation theory is a powerful and flexible tool to design and study groups with various prescribed properties. It will play a key role in our journey among monster groups. There are various approaches to small cancellation theory. Let us first sketch briefly the classical setting. For more details we refer the reader to Lyndon-Schupp [108].

1.1.1 Classical small cancellation theory.

Let $\mathbf{F}(S)$ be the free group generated by a finite set S. Let R be a set of words over the alphabet $S \cup S^{-1}$. The goal is to study the quotient $\bar{G} = \mathbf{F}(S)/\langle\langle R\rangle\rangle$, where $\langle\langle R\rangle\rangle$ stands for the normal subgroup of $\mathbf{F}(S)$ generated by R. We assume that the elements of R are non-trivial and cyclically reduced and denote by R^* the set of all cyclic conjugates of elements of $R \cup R^{-1}$.

A *piece* is a common prefix of two distinct elements in R^*. In other words, a piece is a subword that could cancel when forming the product rs of two elements $r, s \in R^*$ such that $r \neq s^{-1}$. Let $\lambda > 0$. One says that R satisfies the *small cancellation assumption* $C'(\lambda)$ if for all pieces u, for all relations $r \in R$ containing u, we have $|u| < \lambda|r|$. We sometimes make an abuse of terminology and say that the quotient $\bar{G}$ satisfies the $C'(\lambda)$ condition. The $C'(\lambda)$ condition is very flexible and small cancellation groups abound: from a statistical point of view, most of the groups are small cancellation groups, see Ol'shanskiĭ [119]. For many applications, one can use a stronger condition called $C''(\lambda)$. It requires that $|u| < \lambda|r|$ for all pieces u, and all relations $r \in R$ (not necessarily containing u). When the relations in R have approximately the same length, the $C''(\lambda)$ and $C'(\lambda)$ conditions are very similar. Surface groups of genus g are typical examples of small cancellation groups. They admit a presentation satisfying the $C'(\lambda)$ condition, provided $\lambda > 1/4g$. More generally small cancellation theory is an important source of hyperbolic groups.

Theorem 1.1 ([76, Appendix]). *Let S be a finite set and R a finite subset of $\mathbf{F}(S)$. If R satisfies the $C'(1/6)$ condition, then the quotient $\bar{G} = \mathbf{F}(S)/\langle\langle R\rangle\rangle$ is hyperbolic.*

This can be derived from the fact that finitely presented small cancellation groups satisfy a linear isoperimetric inequality.

1.1.2 A geometric point of view.

The origin of small cancellation theory goes back to the work of Dehn, who exploited negative curvature to solve decision problems in surface groups [70]. Nevertheless as Gromov wrote

> *"In the course of the development, the geometric roots were forgotten, and the role of curvature was reduced to a metaphor. (Algebraists do not trust geometry.)"* [84].

Reconnecting with geometry, Gromov explains how to make any $C''(1/6)$ group act on a CAT(-1) space. This approach – later developed by Delzant-Gromov [71], Coulon [2, 5, 6] and Dahmani-Guirardel-Osin [67] – generalizes to the class of groups acting on a hyperbolic space. This will be our main tool in this chapter.

Context. Let us fix a few notations and recall some vocabulary. Let X be a metric length space. If it exists, we write $[x, y]$ for a geodesic joining the points $x, y \in X$. The Gromov product of three points $x, y, z \in X$ is

$$\langle x, y \rangle_z = \frac{1}{2}\left[d(x,z) + d(y,z) - d(x,y) \right].$$

We assume that X is δ-hyperbolic, that is for every $x, y, z, t \in X$, we have

$$\langle x, z \rangle_t \geqslant \min\left\{ \langle x, y \rangle_t, \langle y, z \rangle_t \right\} - \delta.$$

For a comprehensive introduction to hyperbolic geometry we refer the reader to Gromov's seminal article [82] or [60, 76]. We denote by ∂X its boundary at infinity. A subset $Y \subset X$ is *α-quasi-convex*, if $d(x, Y) \leqslant \langle y, y' \rangle_x + \alpha$, for every $x \in X$, $y, y' \in Y$. Given $d \in \mathbf{R}_+$, we denote by Y^{+d} the d-neighborhood of Y. If Y is α-quasi-convex, then $Y^{+\alpha}$ is 2δ-quasi-convex. Thus most of our quasi-convex subsets will be 2δ-quasi-convex.

The *translation length* and *stable translation length* of an isometry g of X are

$$\|g\| = \inf_{x \in X} d(gx, x) \quad \text{and} \quad \|g\|^\infty = \lim_{k \to \infty} \frac{1}{k} d\left(g^k x, x\right).$$

Such an isometry is *loxodromic* if its orbit has exactly two accumulation points in ∂X, that we denote by g^- and g^+. In this case, one associates to g an *axis* $A_g \subset X$. It is a 2δ-quasi-convex $\langle g \rangle$-invariant subset that is quasi-isometric to $\mathbf{R}$ and on which g roughly acts by translation of length $\|g\|$. The exact definition differs from author to author. One option is to define A_g as a suitable neighborhood of all (local quasi-)geodesics of X joining g^- to g^+. In particular, if X is an $\mathbf{R}$-tree, then A_g is simply the bi-infinite geodesic joining g^- to g^+.

The action of a group G on X is *elementary* if its limit set in ∂X contains at most two points.

The small cancellation assumption. Let G be a group acting by isometries on X. We consider a family $\mathcal{Q}$ of pairs (H, Y), where $Y \subset X$ is a 2δ-quasi-convex subset and $H \subset G$ is a subgroup stabilizing Y. In addition, we suppose that $\mathcal{Q}$ is invariant for the action of G given by $g(H, Y) = (gHg^{-1}, gY)$, for every $g \in G$ and $(H, Y) \in \mathcal{Q}$. We call $\mathcal{Q}$ a *relation family.* The goal is to study the quotient $\bar{G} = G/K$, where K is the (normal) subgroup generated by all subgroups H, where (H, Y) runs over $\mathcal{Q}$. To that end, we associate to $\mathcal{Q}$ two parameters that respectively play the role of the length of the longest piece and the length of the shortest relation.

$$\begin{aligned}\Delta(\mathcal{Q}, X) &= \sup\left\{\operatorname{diam}\left(Y_1^{+5\delta} \cap Y_2^{+5\delta}\right) \;:\; (H_1, Y_1) \neq (H_2, Y_2) \in \mathcal{Q}\right\},\\ T(\mathcal{Q}, X) &= \inf\left\{\|h\| \;:\; h \in H \setminus \{1\}, (H, Y) \in \mathcal{Q}\right\}.\end{aligned}$$

Definition 1.2. Let $\lambda \in (0,1)$ and $\varepsilon \in \mathbf{R}_+^*$. We say that $\mathcal{Q}$ satisfies the $C''(\lambda, \varepsilon)$-condition, if

$$\frac{\Delta(\mathcal{Q}, X)}{T(\mathcal{Q}, X)} \leqslant \lambda \quad \text{and} \quad \frac{\delta}{T(\mathcal{Q}, X)} \leqslant \varepsilon.$$

In practice we will be interested in situations where both λ and ε are very small. Delzant and Gromov even speak of *very small cancellation theory* [71].

Example 1.1. Assume that $G = \mathbf{F}(S)$ is the free group generated by a finite set S and X its Cayley graph with respect to S. Let R be a set of words over the alphabet $S \cup S^{-1}$, which is cyclically reduced and invariant under taking inverses. Consider the following relation family

$$\mathcal{Q} = \left\{\left(u \langle r \rangle u^{-1}, uA_r\right) \;:\; r \in R,\ u \in G\right\}.$$

Then $\Delta(\mathcal{Q}, X)$ is exactly the length of the longest piece of R, while $T(\mathcal{Q}, X)$ is the length of the shortest relation. Recall that a tree is 0-hyperbolic. Thus given $\lambda \in (0,1)$, the family $\mathcal{Q}$ satisfies the classical $C''(\lambda)$ condition if and only if it satisfies the $C''(\lambda, \varepsilon)$ condition for some (hence any) $\varepsilon \in \mathbf{R}_+^*$. □

The next example is based on acylindricity. Acylindricity was introduced by Sela (in the context of trees) [145] and extended by Bowditch [47] as follows.

Definition 1.3. Let $D, M\colon \mathbf{R}_+ \to \mathbf{R}_+$ be two functions. The action of G on X is (D, M)-*acylindrical* if for every $r \in \mathbf{R}_+$, for every subset $S \subset G$, if the diameter of

$$\{x \in X \;:\; d(sx, x) \leqslant r,\ \forall s \in S\}$$

is larger than $D(r)$, then S contains at most $M(r)$ elements. The action is *acylindrical* if there exist two functions, $D, M\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that it is (D, M)-acylindrical.

Actually, if X is δ-hyperbolic, it suffices to check the following property: there exist $D_0, M_0 \in \mathbf{R}_+$ such that for every subset $S \subset G$, if the diameter of

$$\{x \in X \;:\; d(sx, x) \leqslant 4000\delta,\ \forall s \in S\}$$

is larger than D_0, then S contains at most M_0 elements [67].

Groups acting acylindrically on hyperbolic spaces provide a far-reaching generalization of hyperbolic groups which has been very fruitful in the last decades, see for instance Osin [125]. A classical example is the mapping class group of a closed surface acting on the corresponding curve complex [47].

Two loxodromic elements $g_1, g_2 \in G$ are *commensurable*, if there exist $n_1, n_2 \in \mathbf{Z}$ such that $g_1^{n_1}$ and $g_2^{n_2}$ are conjugate.

Example 1.2. Let $D, M\colon \mathbf{R}_+ \to \mathbf{R}_+$ be two functions. There is $p \in \mathbf{N} \setminus \{0\}$ such that for every $\lambda \in (0,1)$ and $\varepsilon \in \mathbf{R}_+^*$, there exists $N \in \mathbf{N}$, with the following property. Let G be a group acting by isometries on a hyperbolic length space X. Assume that the action is (D,M)-acylindrical. Let $R = \{g_1, \dots, g_r\}$ be a finite collection of pairwise non-commensurable loxodromic elements. Let $n_1, \dots, n_r \in \mathbf{N}$. Consider the family

$$\mathcal{Q} = \left\{ \left(u \left\langle g_i^{pn_i} \right\rangle u^{-1}, uA_{g_i} \right) \ :\ u \in G,\ i \in [\![1,r]\!] \right\}.$$

If $n_i \geqslant N$, for every $i \in [\![1,r]\!]$, then $\mathcal{Q}$ satisfies the $C''(\lambda,\varepsilon)$ condition. The proof relies on the following two consequences of acylindricity.

1. There exists $\varepsilon \in \mathbf{R}_+^*$ (which only depends on D and M) such that for every loxodromic element $g \in G$, we have $\|g\|^\infty > \varepsilon$, see Bowditch [47]. This provides a lower bound on the length of relations.
2. The upper bound on the length of pieces follows from the following fact. If $g, h \in G$ are loxodromic elements which do not generate an elementary subgroup then

$$\operatorname{diam}\left(A_g^{+5\delta} \cap A_h^{+5\delta}\right) \leqslant M(1000\delta)\,\|h\| + \|g\| + D(1000\delta) + 1000\delta.$$

The proof of the latter inequality is quite a standard argument [67]. Recall that g roughly acts on A_g by translation of length $\|g\|$. Same with h. Focusing on the overlap between A_g and A_h we see that the commutator $[g, h^j]$ almost fixes a segment of length D, for all $j \in \mathbf{N}$ satisfying

$$j\,\|h\| + \|g\| + D + 1000\delta \leqslant \operatorname{diam}\left(A_g^{+5\delta} \cap A_h^{+5\delta}\right)$$

see Figure 1.1. If the intersection of A_g and A_h is very long (compare to the translation lengths of g and h) then by acylindricity two such commutators must coincide, which forces g and h to generate an elementary subgroup. □

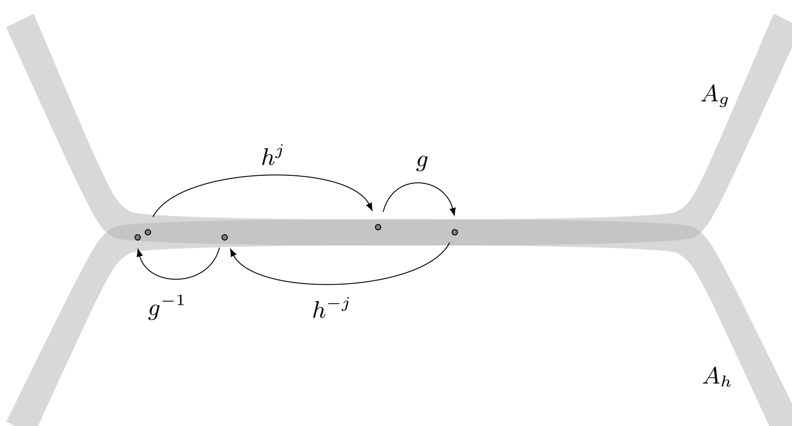


Figure 1.1 – Axes with a long overlap

The framework of geometric small cancellation theory is much wider than these two examples. In particular, it handles relation families $\mathcal{Q}$, where the subgroups H are not necessarily cyclic, see Coulon [2] or Arzhantseva-Delzant [32]. It covers, among others, the theory of small cancellation over free products [108], the graphical small cancellation theory [85], etc. Let us state a first version of the small cancellation theorem that echoes Theorem 1.1.

Theorem 1.4 (Simplified small cancellation theorem)**.** *For every $D \geqslant 0$, there exist $\lambda \in (0,1)$ and $\varepsilon \in \mathbf{R}_+^*$ with the following property. Let X be a δ-hyperbolic length space endowed with an action by isometries of a group G and $\mathcal{Q}$ be a relation family. Let K be the normal subgroup of G generated by all the subgroups H where (H,Y) runs over $\mathcal{Q}$. Let $\bar{G} = G/K$ and $\pi \colon G \twoheadrightarrow \bar{G}$ the corresponding projection.*

If $\mathcal{Q}$ satisfies the $C''(\lambda, \varepsilon)$ condition, then there exists a hyperbolic length space $\bar{X}$ endowed with an action by isometries of $\bar{G}$. Moreover,

1. *for every $(H,Y) \in \mathcal{Q}$, the map π induces an embedding from $\operatorname{Stab}(Y)/H$ into $\bar{G}$;*
2. *for every $x \in X$, the map π is one-to-one when restricted to the set*

$$\{g \in G \ : \ d\left(gx, x\right) \leqslant D\delta\}.$$

Notice that the statement makes no serious assumption on the action of G on X. The true hypothesis is the *existence* of a relation family satisfying the $C''(\lambda, \varepsilon)$ condition.

Saying that $\bar{G}$ acts on a hyperbolic space without further details is not very helpful. Indeed, every group admits an action by isometries on a hyperbolic space, namely the trivial one! The general philosophy though is that the action of $\bar{G}$ will inherit properties from the action of G. Suppose for instance that the action of G on X is proper and co-compact. Under the assumptions of Theorem 1.4, the action of $\bar{G}$ on $\bar{X}$ is in general not proper and co-compact. This happens indeed if $\mathcal{Q}/G$ is infinite or if $\operatorname{Stab}(Y)/H$ is infinite for some $(H,Y) \in \mathcal{Q}$. However, if we forbid these obvious obstructions, then one shows that the action of $\bar{G}$ on $\bar{X}$ is proper and co-compact. In particular, the quotient $\bar{G}$ is hyperbolic, which generalizes Theorem 1.1. Similarly, if the action of G on X is acylindrical, then so will be the one of $\bar{G}$ on $\bar{X}$, see [67].

The geometry of the space $\bar{X}$ is also very convenient to study the group $\bar{G}$. Even if $\bar{G}$ is a hyperbolic group, it is designed in such a way that its geometry is finer than the one of the Cayley graph of $\bar{G}$. To make this idea more precise let us review first its construction.

The cone-off space and its quotient. Fix a parameter $\rho \in \mathbf{R}_+^*$. Its value will be made precise later (see Theorem 1.5). It should be thought of as a very large radius. Given $(H,Y) \in \mathcal{Q}$, the *cone of radius ρ over Y*, denoted by $Z_\rho(Y)$ or simply $Z(Y)$, is the quotient of $Y \times [0\,,\rho]$ by the equivalence relation that identifies all the points of the form $(y,0)$. The class v of $(y,0)$ is called the *apex* of the cone. Such a cone comes with a natural embedding $Y \to Z(Y)$ sending y to (y,ρ). We endow $Z(Y)$ with a metric modeled on the hyperbolic space $\mathbf{H}^2$: given two points $x = (y,r)$ and $x' = (y',r')$ in $Z(Y)$ we let

$$\operatorname{ch} d_{Z(Y)}(x,x') = \operatorname{ch} r \operatorname{ch} r' - \operatorname{sh} r \operatorname{sh} r' \cos\theta(y,y'),$$

where

$$\theta(y,y') = \min\left\{\pi, \frac{d(y,y')}{\pi \operatorname{sh}\rho}\right\},$$

see [51]. This distance has the following geometric interpretation. Consider a comparison triangle in the hyperbolic plane $\mathbf{H}^2$ such that the lengths of two sides are respectively r and r' and the angle between them is $\theta(y,y')$. According to the law of cosines, $d(x,x')$ is exactly the length of the third side of the triangle (see Figure 1.2).

Example 1.3. If Y is a circle whose perimeter is $2\pi \operatorname{sh}\rho$ endowed with the length metric, then $Z(Y)$ is the closed hyperbolic disc of radius ρ. If Y is the real line, then $Z(Y) \setminus \{v\}$ is the universal cover of the punctured hyperbolic disc of radius ρ. □

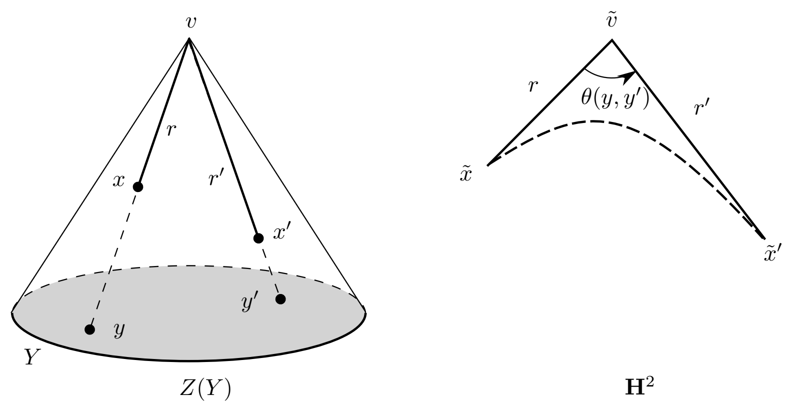


Figure 1.2 – Geometric interpretation of the distance in the cone $Z_\rho(Y)$.

The *cone-off of radius* ρ *over* X *relative to* $\mathcal{Q}$ is the space $\dot{X}_\rho(\mathcal{Q})$, or simply $\dot{X}$, obtained by attaching for each pair $(H, Y) \in \mathcal{Q}$, the cone $Z_\rho(Y)$ on X along Y with the map $Y \to Z_\rho(Y)$. We endow $\dot{X}$ with the largest metric such that the maps $X \to \dot{X}$ and $Z_\rho(Y) \to \dot{X}$ are 1-Lipschitz. We denote by $\mathcal{V}$ the set of all cone apices in $\dot{X}$.

Remark. In order to reduce the level of technicality, we oversimplified the construction. The metric space $\dot{X}$ defined above may not be a length space, which can be a source of numerous complications. One can turn $\dot{X}$ into a length space by slightly perturbing the metric, see [5]. From now on though, we will make as if our simplified construction is a length space. This is in essence correct and will not affect the rest of the exposition. □

The relation family $\mathcal{Q}$ is G-invariant, hence the action of G on X extends to an action by isometries of G on $\dot{X}$. We denote by $\bar{X}_\rho(\mathcal{Q})$, or simply $\bar{X}$, the quotient $\bar{X}_\rho(\mathcal{Q}) = \dot{X}_\rho(\mathcal{Q})/K$. If x is a point in $\dot{X}$, we write $\bar{x}$ for its image in $\bar{X}$. We endow $\bar{X}$ with the quotient pseudo-metric inherited from $\dot{X}$. The group $\bar{G}$ naturally acts on $\bar{X}$ by isometries.

Example 1.4. In the setting of Example 1.1, the space $\bar{X}$ is topologically the Cayley complex of the presentation $\langle S | R \rangle$. Nevertheless the additional metric structure on $\bar{X}$ plays an important role. □

Small cancellation theorem The main statement of geometric small cancellation theory can now be stated as follows, see [71, 5].

Theorem 1.5 (Small cancellation theorem)**.** *There exist $\delta_0, \delta_1, \Delta_0, \rho_0 \in \mathbf{R}_+^*$ with the following properties.*

Let G be a group acting by isometries on a δ-hyperbolic length space X. Let $\mathcal{Q}$ be a relation family. Let K be the normal subgroup of G generated by all the subgroups H where (H, Y) runs over $\mathcal{Q}$. Let $\bar{G} = G/K$.

Let $\rho \geqslant \rho_0$. Assume that $\delta \leqslant \delta_0$, $\Delta(\mathcal{Q}, X) \leqslant \Delta_0$, and $T(\mathcal{Q}, X) \geqslant 10\pi \operatorname{sh} \rho$. Then the following holds.

1. *The space $\bar{X} = \bar{X}_\rho(\mathcal{Q})$ is a δ_1-hyperbolic length space endowed with an action by isometries of $\bar{G}$.*

2. *Let $(H, Y) \in \mathcal{Q}$. Let $\bar{v}$ be the image in $\bar{X}$ of the apex v of $Z_\rho(Y)$. The projection $G \to \bar{G}$ induces an isomorphism from* $\mathrm{Stab}(Y)/H$ *onto* $\mathrm{Stab}(\bar{v})$.
3. *For every $r \in (0, \rho/20]$, for every $x \in \dot{X}$, if $d(x, \mathcal{V}) \geqslant 2r$, then the projection $\dot{X} \to \bar{X}$ induces an isometry from $B(x, r)$ onto $B(\bar{x}, r)$.*
4. *For every $x \in \dot{X}$, and $g \in K \setminus \{1\}$, we have $d_{\dot{X}}(gx, x) \geqslant \min\{2r, \rho/5\}$, where $r = d(x, \mathcal{V})$. In particular, K acts freely on $\dot{X} \setminus \mathcal{V}$. Moreover, the projection $\dot{X} \to \bar{X}$ induces a covering map $\dot{X} \setminus \mathcal{V} \to \bar{X} \setminus \bar{\mathcal{V}}$.*

Remark. Note that our assumption on $\mathcal{Q}$ takes an absolute form here, whereas the $C''(\lambda, \varepsilon)$ condition is a relative one. Nevertheless Theorem 1.4 is obtained by first rescaling the space X and then applying Theorem 1.5. □

We wrote previously that the geometry of $\bar{X}$ is somewhat finer that the one of the Cayley graph of $\bar{G}$. Let us to mention some features to support this claim.

— The parameters $\delta_0, \delta_1, \Delta_0$, and ρ_0 do not depend on G or X, but only on the small cancellation condition satisfied by $\mathcal{Q}$. For many applications, δ_0 and Δ_0 (respectively ρ_0) can be chosen arbitrarily small (respectively large). In practice, we fix them so that

$$\max\{\delta_0, \Delta_0\} \ll \delta_1 \ll \rho_0 \leqslant \rho \ll \pi \operatorname{sh} \rho.$$

In particular, the hyperbolicity constant δ_1 of $\bar{X}$ is much smaller than the injectivity radius of the map $X \to \bar{X}$, which is at least $\rho/10$ by (4). For comparison, in the setting of Example 1.1, the injectivity radius of the projection $\mathbf{F}_r \twoheadrightarrow \bar{G}$ is roughly the hyperbolicity constant of $\bar{G}$ (endowed with the word metric).

— One can think of $\dot{X}$ as a space with a thin/thick decomposition. Recall that $\mathcal{V}$ stands for the set of apices in $\dot{X}$. The thin parts of $\dot{X}$ are the balls of radius $99\rho/100$ centered at apices in $\mathcal{V}$, while the thick part is the complement of those balls. The space $\bar{X}$ inherits from this thin/thick decomposition. Point (3) states that the map $\dot{X} \twoheadrightarrow \bar{X}$ induces a local isometry from the thick part of $\dot{X}$ onto its image.

These features make the geometry of $\bar{X}$ particularly efficient to study $\bar{G}$. Let us explain this idea with an example.

Example 1.5. We claim that every finite subgroup $\bar{F}$ of $\bar{G}$ is either the isomorphic image of a finite subgroup F of G or comes in an "obvious" way from the relation family. Indeed, since $\bar{F}$ is elliptic, it has an orbit $\bar{F}\bar{x} \subset \bar{X}$ of diameter at most $10\delta_1$. We distinguish two cases. Assume first that $\bar{F}\bar{x}$ is contained in the thick part of $\bar{X}$. By Theorem 1.5 (3) and (4) there exist a subgroup F of G and a point x in the thick part of $\dot{X}$, such that the projection $G \twoheadrightarrow \bar{G}$ induces an isomorphism from F onto $\bar{F}$ while the map $\dot{X} \to \bar{X}$ induces an isometry from Fx onto $\bar{F}\bar{x}$. In particular, $\bar{F}$ is the image of a finite subgroup of G. Suppose now that $\bar{F}\bar{x}$ intersects the thin part. Using the triangle inequality, we observe that $\bar{F}$ fixes an apex $\bar{v} \in \bar{\mathcal{V}}$ (two apices are at a distance at least 2ρ far apart). Hence by Theorem 1.5 (2) $\bar{F}$ is a subgroup of $\mathrm{Stab}(Y)/H$, for some relation $(H, Y) \in \mathcal{Q}$, which completes the proof of our claim. As a consequence, if G is torsion-free and $H = \mathrm{Stab}(Y)$ for every $(H, Y) \in \mathcal{Q}$, then $\bar{G}$ is torsion-free as well. □

This example illustrates a more general philosophy. Hyperbolic geometry abounds of local-to-global phenomena. It is very common that a global property can be read off at a small scale. By small scale we mean that it involves a subset $\bar{Y} \subset \bar{X}$ whose diameter is at most $1000\delta_1$ say. In this situation, we use the thin/thick decomposition, together with the fact that $\delta_1 \ll \rho$. If $\bar{Y}$ lies in the thick part, we isometrically lift it into $\dot{X}$, and exploit the geometry of the original group G. If $\bar{Y}$ intersects the thin part, we can often reduce the problem to some properties of $\mathrm{Stab}(Y)/H$.

Some applications. Since its origin small cancellation has been a very fruitful tool. We mention here a few original applications drawn from our work.

- In [2], we built new examples of aspherical polyhedra. Those are obtained by coning off a totally geodesic real sub-manifold of a complex hyperbolic manifold.
- The Farrell-Jones conjecture is a group property that aims to describe the algebraic K-theory of its group ring in terms of the K-theories of simpler group rings. It has outstanding consequences such as the Novikov conjecture, the Borel conjecture, the Kaplansky conjecture, and the Serre conjecture. Generalizing the coarse geodesic flow method that was used for hyperbolic groups, Bartels proved that if a group G is hyperbolic relative to parabolic subgroups satisfying the Farrell-Jones conjecture, then G also satisfies the conjecture [36]. With Yago Antolin and Giovanni Gandini, we used small cancellation theory to give an alternative proof of this theorem, when the parabolic subgroups are residually finite [1].
- It is a famous open question whether every hyperbolic group is residually finite. Recall that a group is hyperbolic if and only if it acts properly co-compactly on a hyperbolic space. We wondered whether the question has a (non-trivial) answer if we remove the co-compactness assumption. Every countable group admits a proper action on a hyperbolic space, namely the parabolic action on a combinatorial horoball [86]. Thus, to obtain an interesting class of groups, we have to strengthen the properness assumption. We say that a group G acts *uniformly properly* on X if for every $r \in \mathbf{R}_+$, there exists $N \in \mathbf{N}$, such that for all $x \in X$,

  $$|\{g \in G \ : \ d(x, gx) \leqslant r\}| \leqslant N.$$

  In particular, a uniformly proper action is acylindrical. With Denis Osin we made a new addition to the bestiary of groups, and proved that there exists a finitely generated group G with a non-elementary, uniformly proper action on a hyperbolic space, such that every amenable quotient of G is trivial [24]. In particular, G is not residually finite.

1.2 Infinite periodic groups

We now explain how small cancellation can be use to tackle the Burnside problem. In what follows we write $\mathbf{D}_\infty$ (respectively $\mathbf{D}_n$) for the infinite dihedral group (respectively the dihedral group of order $2n$).

1.2.1 Different flavors of periodic groups

Quotient of hyperbolic groups. Let n be an integer and G be a group. Recall that G^n stands for the (normal) subgroup of G generated by the n-th power of all its elements. Our goal is to investigate periodic quotients of the form G/G^n provided G has some kind of negative curvature. Using the geometric approach of small cancellation theory (as described in Section 1.1) Delzant and Gromov revisited the Burnside problem. More precisely, they proved the following statement originally due to Ol'shanskiĭ.

Theorem 1.6 (Ol'shanskiĭ [118], Delzant-Gromov [71])**.** *Let G be a non-elementary, torsion-free hyperbolic group. There exists a critical exponent $N \in \mathbf{N}$, such that for every odd integer $n \geqslant N$, the quotient G/G^n is infinite.*

Unfortunately the work of Delzant and Gromov only applies to *odd* exponents. Ivanov [94] and Lysenok [109] have proved that $\mathbf{B}_r(n)$ is also infinite for sufficiently large even exponents.

However this case is much more complicated to handle. In a series of articles [5, 6, 9] we extend the Delzant-Gromov approach to build new examples of periodic groups and finally tackle the case of free Burnside groups of *even* exponents. Actually, we give a different proof of the following result by Ivanov and Ol'shanskiĭ.

Theorem 1.7 (Ivanov-Ol'shanskiĭ [95], Coulon [9]). *Let G be a non-elementary hyperbolic group. There exist $p, N \in \mathbf{N}$, such that for every integer $n \geqslant N$, which is a multiple of p, the quotient G/G^n is infinite.*

Remark 1.2.1. Unlike in Theorem 1.6, it is not assumed that G is torsion-free. It is thus important to consider exponents which are multiple of a fixed integer p. Assume indeed that $G = \mathbf{Z}/3\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$. Then for every integer $n \in \mathbf{N}$ which is co-prime with 3, the quotient G/G^n is trivial. For the theorem to holds, n better be a multiple of the order of any elliptic element in G. This is not enough though. Here is an example.

Consider first a non-elementary hyperbolic group A with the following properties: non-trivial finite order elements in A have order 3; A is the normal closure of a single finite order element $a \in A$. Such a group can be obtained as a small cancellation quotient of $\mathbf{Z}/3\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$. Consider also the semi-direct product $B = C \rtimes \mathbf{Z}$, where $C = \mathbf{Z}/3\mathbf{Z}$ and the generator t of $\mathbf{Z}$ acts on C by the automorphism sending every element to its inverse. Form now the amalgamated product $G = A *_C B$ where C is identified with $\langle a \rangle \subset A$. The group G is hyperbolic, non-elementary, and all its non-trivial finite order elements have order 3. A direct computation shows that for every $c \in C$, for every $n \in \mathbf{N}$, we have

$$(ct)^n = c\left(tct^{-1}\right)\left(t^2ct^{-2}\right)\cdots\left(t^{n-1}ct^{-n+1}\right)t^n$$

If n is odd, we get $(ct)^n t^{-n} = c$. It follows that the image of C in G/G^n is trivial. Consequently the same holds for $\langle a \rangle$ and thus A. Hence $G/G^n = \mathbf{Z}/n\mathbf{Z}$ is finite. For Theorem 1.7 to work, we need n be a multiple of 6. The reason is that C is normalized by an element (here t) inducing an automorphism of C with order 2. □

As already observed by Ol'shanskiĭ and Ivanov [95], one can elaborate on the structure of G/G^n. For instance, under the assumptions of Theorem 1.7,

1. the word problem and the conjugacy problem in G/G^n are solvable;
2. every finite subgroup of G/G^n embeds in a direct product of the form $F \times \mathbf{D}_n \times \mathbf{D}_{n_2} \times \cdots \times \mathbf{D}_{n_2}$, where F is a finite subgroup of G and n_2 the largest power of 2 dividing n;
3. if U is a finite subset of G, then the projection $G \twoheadrightarrow G/G^n$ is one-to-one when restricted to U, provided n is a sufficiently large multiple of p.

Beyond hyperbolicity. Our strategy covers a larger class of groups than just hyperbolic ones. We understand indeed (partially) periodic quotients of any group G as soon as it enjoys a "controlled" action on a hyperbolic space. More precisely, Theorem 1.7 is a particular case of the following new result.

Theorem 1.8 (Coulon [9]). *Let G be a group with a non-elementary acylindrical action on a hyperbolic length space X. Let $r \in \mathbf{R}_+^*$. There exist $p, N \in \mathbf{N}$ with the following properties. For every integer $n \geqslant N$ which is a multiple of p, there is a quotient Q_n of G such that the following holds.*

1. *For every $x \in X$, the projection $\pi\colon G \twoheadrightarrow Q_n$ is one-to-one when restricted to*

$$\{g \in G \ : \ d(gx, x) < r\}.$$

In particular, π is one-to-one when restricted to any elliptic subgroup of G (for its action on X) and every non-trivial element of $\ker \pi$ is loxodromic.

2. *For every $q \in Q_n$, either q is the image of an elliptic element of G or $q^n = 1$. Moreover the projection $G \twoheadrightarrow G/G^n$ factors through π. Consequently, if every elliptic subgroup of G is periodic with exponent n, then $Q_n = G/G^n$.*

3. *$\ker \pi$ is not normally generated by a finite set.*

Note that if G contains an infinite elliptic subgroup, then Q_n is automatically infinite by (1). Unlike in Theorem 1.7, saying that Q_n is infinite is not very conclusive. This is the purpose of (3). It states that Q_n is not obtained from G by adding only finitely many relations.

Remark 1.2.2. As stated, the integers p and N seem to depend on the group G, the space X and the distance r. We made this choice to keep the exposition less technical. However one has a precise control on these parameters. Essentially, p only depends on the finite subgroups of G and their automorphisms. This is a reminiscent of the phenomena stressed in Remark 1.2.1. In particular, if G is torsion-free, the theorem applies for all integers $n \geqslant N$ which are either odd [6] or multiple of 128 [9]. Similarly the critical exponent N only depends on r and the functions $D, M\colon \mathbf{R}_+ \to \mathbf{R}_+$ controlling the acylindricity of G (see Definition 1.3). □

We discuss now two applications of Theorem 1.8, the first has a geometric flavor, while the second is more algebraic.

Partially periodic quotient of mapping class groups. Let Σ be a compact surface of genus g with k boundary components. In the rest of this paragraph we assume that $3g + k - 3 > 1$. The *mapping class group* $\mathrm{Mod}(\Sigma)$ is the group of orientation preserving self-homeomorphisms of Σ, defined up to isotopy. A mapping class $f \in \mathrm{Mod}(\Sigma)$ is

1. *periodic*, if it has finite order;
2. *reducible*, if it permutes a collection of essential non-peripheral curves (up to isotopy);
3. *pseudo-Anosov*, if there exists a homeomorphism in the class of f that preserves a pair of transverse foliations and rescale them in an "appropriate" way.

It follows from Thurston's work that any element of $\mathrm{Mod}(\Sigma)$ falls into one these three categories [153]. The *curve complex* X is a simplicial complex associated to Σ, first introduced by Harvey [91]. A d-simplex of X is a collection of $d+1$ homotopy classes of curves of Σ that can be disjointly realized. Masur and Minsky proved that X is hyperbolic [110]. By construction, X is endowed with an action by isometries of $\mathrm{Mod}(\Sigma)$, which turns out to be acylindrical, see Bowditch [47]. It provides another characterization of the above classification: a mapping class is periodic or reducible (respectively pseudo-Anosov) if and only if it is elliptic (respectively loxodromic) when acting on the curve complex [110]. Applying Theorem 1.8 to this action yields the following result.

Corollary 1.9 (Coulon [9])**.** *Let Σ be a compact surface of genus g with k boundary components such that $3g+k-3 > 1$. There exist $p, N \in \mathbf{N}$ such that for every integer $n \geqslant N$ which is a multiple of p, there is a quotient Q_n of $\mathrm{Mod}(\Sigma)$ with the following properties.*

1. *If E is a subgroup of $\mathrm{Mod}(\Sigma)$ that does not contain a pseudo-Anosov element, then the projection $\pi\colon \mathrm{Mod}(\Sigma) \twoheadrightarrow Q_n$ induces an isomorphism from E onto its image.*

2. *Let $f \in \mathrm{Mod}(\Sigma)$. Either $f^n = 1$ in Q_n or f coincides in Q_n with a periodic or a reducible element.*
3. $\ker \pi$ *is not normally generated by a finite set. All its non-trivial elements are pseudo-Anosov.*

Ivanov asked if the n-periodic quotient of the mapping class group is infinite [93]. Funar gave a positive answer for surfaces of genus two using quantum representations [75]. In the previous statement the kernel of π is made only of pseudo-Anosov elements. For comparison, here is another result where we "kill" the n-th power of reducible elements of the mapping class groups.

Theorem 1.10 (Coulon [6])**.** *Let Σ a surface of genus $g \geqslant 2$. Let S be the set of all Dehn twists of Σ. There exists $N \in \mathbf{N}$ such that for every odd exponent $n \geqslant N$, the free group $\mathbf{F}_2$ embeds into* $\mathrm{Mod}(\Sigma)/\langle\langle s^n, S \in S\rangle\rangle$.

Amalgamated product in the Burnside variety. Recall that the Burnside variety $\mathfrak{B}_n$ is the class of all groups with exponent n. In order to begin a systematic study of $\mathfrak{B}_n$ one needs to understand basic operations in this variety, such as amalgamation.

Proposition-Definition 1.11. *Let $A_1, A_2 \in \mathfrak{B}_n$. Let $\iota_i\colon C \hookrightarrow A_i$ be two monomorphisms. There exist a group in $\mathfrak{B}_n$, called the* amalgamated n-periodic product of A_1 and A_2 over C *and denoted by $A_1 *^n_C A_2$ and two morphisms $\jmath_i\colon A_i \to A_1 *^n_C A_2$ with the following property. For every group $G \in \mathfrak{B}_n$, for every morphisms $\phi_1\colon A_1 \to G$ and $\phi_2\colon A_2 \to G$ with $\phi_1 \circ \iota_1 = \phi_2 \circ \iota_2$, there exists a unique morphism $\phi\colon A_1 *^n_C A_2 \to G$ such that the following diagram commutes.*

$$\begin{tikzcd}
C \arrow[r, "\iota_1"] \arrow[d, "\iota_2"'] & A_1 \arrow[d, "\jmath_1"] \arrow[ddr, bend left, "\phi_1"] & \\
A_2 \arrow[r, "\jmath_2"] \arrow[drr, bend right, "\phi_2"'] & A_1 *^n_C A_2 \arrow[dr, dashed, "\phi"] & \\
& & G
\end{tikzcd}$$

Proof. It suffices to take for $A_1 *^n_C A_2$ the n-periodic quotient of the (regular) amalgamated product $A_1 *_C A_2$. ■

It follows from the proposition that $A_1 *^n_C A_2$ is unique up to isomorphism. If C is trivial, we recover the notion of free product in the Burnside variety studied by Adian [27]. Unlike for regular amalgamated products, there is no reason though that the maps $\jmath_i\colon A_i \to A_1 *^n_C A_2$ should be injective.

Example 1.6. We view the free group $\mathbf{F}(a, b)$ as the fundamental group of the punctured torus. Let ϕ_1 and ϕ_2 be the Dehn twists given by $\phi_1\colon a \mapsto a, b \mapsto ba$ and $\phi_2\colon a \mapsto ab, b \mapsto b$. Let n be an odd exponent. Note that ϕ_i induces an automorphism of order n of $C = \mathbf{B}_2(n)$ (that we still denote by ϕ_i). Consequently, the semi-direct product $A_i = C \rtimes_{\phi_i} \mathbf{Z}/n\mathbf{Z}$ belongs to $\mathfrak{B}_m$, for $m = n^2$. We denote by t_i the generator of $\mathbf{Z}/n\mathbf{Z}$ acting by conjugation on C like ϕ_i. The product $\phi_1\phi_2$ represents a pseudo-Anosov mapping class of the punctured torus. Hence it induces an infinite order automorphism ψ of C, provided n is large enough, see Coulon [4]. In particular, there is some $c \in C$ such that $\psi^m(c)$ and c are distinct. However, in $A_1 *^m_C A_2$, we get the following relation

$$\psi^m(c) = (t_1t_2)^m c (t_1t_2)^{-m} = c.$$

Thus the natural map $C \to A_1 *^m_C A_2$ is not an embedding. □

Malnormality can be used to overcome this difficulty. A subgroup H of G is *malnormal* if $gHg^{-1} \cap H = \{1\}$, for every $g \in G \setminus H$.

Theorem 1.12 (Coulon [6]). *There exists a critical exponent $N \in \mathbf{N}$, such that for every odd integer $n \geqslant N$, the following holds. Let $A, B \in \mathfrak{B}_n$. Let C be a common subgroup of A and B, that is malnormal in A or B. Then A and B naturally embed in $A *_C^n B$.*

The idea is to apply Theorem 1.8 – or more precisely the version for odd exponents given in [6] – to the action of $G = A *_C B$ on the corresponding Bass-Serre tree. Indeed, this action is 2-acylindrical in the sense of Sela (i.e. the pointwise stabilizer of any subset whose diameter is larger than 2 is trivial). A key point is that no finite subgroup of G is normalized by a loxodromic element. In this way, we avoid pathologies described in Remark 1.2.1 or Example 1.6. It is also important that the critical N in Theorem 1.8 only depends on the acylindricity of G (see Remark 1.2.2). In particular, we can directly study the n-periodic quotient of G, where n is exactly the exponent of A and B. There is no need to increase n or pass to a multiple.

Remark. One can weaken the malnormality assumption, and only suppose that the action of $A *_C B$ on its Bass-Serre tree is k-acylindrical. However the critical exponent N will depend on k. □

1.2.2 Strategy for studying periodic groups

Let us highlight the main steps in the proof of Theorem 1.8. For simplicity we restrict our attention to *free* Burnside groups. This example suffices to underline all the difficulties coming from even torsion.

A sequence of approximation groups. All known strategies for studying free Burnside groups (regardless if the exponent is even or odd) start in the same way: given $n \in \mathbf{N}$, one produces by induction an *approximation sequence* of hyperbolic groups

$$\mathbf{F}_r = G_0 \twoheadrightarrow G_1 \twoheadrightarrow G_2 \cdots \twoheadrightarrow G_k \twoheadrightarrow G_{k+1} \twoheadrightarrow \dots \tag{1.1}$$

whose direct limit is exactly $\mathbf{B}_r(n)$. At each step, G_{k+1} is obtained from G_k by adding new relations of the form $g^n = 1$, where g runs over a set of "small" loxodromic elements of G_k. The crucial point is to prevent this sequence to collapse to a finite group.

Indeed, if $\mathbf{B}_r(n)$ is finite, then in particular it is finitely presented. Thus the sequence (G_k) eventually stabilizes. On the contrary, if we can make sure that each G_k is non-elementary, it would imply that $\mathbf{B}_r(n)$ is infinite. This is achieved by small cancellation arguments. The novelty of our method is to use a geometric point of view *à la* Delzant-Gromov in the context of periodic groups of even exponents. In particular, we attach to each group G_k a preferred hyperbolic space X_k on which the group acts properly and co-compactly.

Working with geometric small cancellation has the following advantage: almost every useful property coming from the relations defining G_k is captured by the structure of X_k. Consequently when studying the quotient map $G_k \twoheadrightarrow G_{k+1}$ one can completely forget the relations defining G_k and rely only on the geometry of X_k. Following Delzant-Gromov [71], this allows us to formulate – unlike in [116, 117, 109, 94] – the induction hypothesis used to build the approximation sequence (1.1) in a rather compact form.

Concretely the induction works as follows. First we fix "very small" parameters $\lambda, \varepsilon \in \mathbf{R}_+^*$. Assume that the group G_k and the space X_k have been already defined. Let δ_k be the hyperbolicity constant of X_k. Define G_{k+1} as the quotient of G_k by the following relation family

$$\mathcal{Q}_k = \{(\langle g^n \rangle, A_g) \ : \ g \in G_k, \ \|g\| \leqslant 10\delta_k\}.$$

Example 1.2 indicates that $\mathcal{Q}_k$ satisfies the $C''(\lambda, \varepsilon)$ assumption, provided n is larger than some critical exponent $N_k \in \mathbf{N}$. For such an exponent n, we build the space X_{k+1} on which G_{k+1} is acting using Theorem 1.4. The difficulty is to make sure that *at each step* we can choose the same exponent n so that $\mathcal{Q}_k$ satisfies the $C''(\lambda, \varepsilon)$ condition. The critical exponent N_k given in Example 1.2 only depends on the acylindricity parameters of the action of G_k. If we had a uniform control on these parameters along the sequence (1.1), then we could safely iterate the process. This is exactly how things work when n is *odd*. For *even* exponent, we face a thorny problem. To explain it, let us start with a toy example.

Example 1.7. Assume that G is a hyperbolic group containing a subgroup of the form $H = (\mathbf{D}_\infty * \mathbf{Z}) \times F$, where F is a finite subgroup. The dihedral factor is generated by two elements of order two, say s and t. Let $g_1, g_2, g_3 \in H$ be loxodromic elements and set $t_i = g_i t g_i^{-1}$. As H is non-elementary, we can choose them so that s and t_i generate an infinite dihedral group, while st_1, st_2, and st_3 are pairwise non-commensurable. Consider the hyperbolic quotient

$$\bar{G} = G / \langle\!\langle (st_1)^n, (st_2)^n, (st_3)^n \rangle\!\rangle.$$

When n is even, s and t_i generate a subgroup of $\bar{G}$ isomorphic to $\mathbf{D}_n$ in which $(st_i)^{n/2}$ is central. In particular, s commutes with the involution $u_i = s(st_i)^{n/2}$. For a suitable choice of g_1, g_2 and g_3, one can prove that u_1, u_2 and u_3 generate a non-elementary subgroup $L \subset \bar{G}$. In particular, there exists $h \in L$ such that $\langle u_1, u_2, h \rangle \cong \mathbf{D}_\infty * \mathbf{Z}$. We have seen that L commutes with s. By construction, it also commutes with (the image of) F. Hence $\bar{G}$ contains a subgroup $\bar{H}$ isomorphic to $(\mathbf{D}_\infty * \mathbf{Z}) \times F \times \mathbf{Z}/2$. □

Let us now go back to the approximation sequence (1.1). The first group is $G_0 = \mathbf{F}_r$. Since n is even, the second group G_1 contains an element of order 2, hence a subgroup isomorphic to $\mathbf{D}_\infty * \mathbf{Z}$. Starting from there and iterating the above example, we see that the approximation group G_k contains subgroups of the form $\mathbf{D}_\infty \times (\mathbf{Z}/2\mathbf{Z})^m$ for an arbitrary value of m (provided k is large enough). Since G_k is hyperbolic, every elementary subgroup $E \subset G_k$ is virtually cyclic. Nevertheless, the index in E of the maximal infinite cyclic subgroup cannot be bounded independently of k. Hence it is impossible to uniformly control the acylindricity parameters of the action of G_k on X_k.

A Margulis' lemma. For the induction to work, we need a weaker version of acylindricity. More precisely, we use three numerical invariants. Consider a group G acting by isometries on a δ-hyperbolic space X.

1. $A(G, X)$ is characterized as follows: given $S \subset G$, if the set
$$\{x \in X \ : \ d(sx, x) \leqslant 4000\delta, \ \forall s \in S\}$$
has diameter larger that $A(G, X)$, then S generates an elementary subgroup. (Note, for comparison, that standard acylindricity would require S to be finite.)
2. $\nu(G, X)$ is the smallest integer m with the following property: let $g, h \in G$ with h loxodromic. If g, hgh^{-1}, h^2gh^{-2}, ..., $h^m g h^{-m}$ generate an elementary subgroup, then so do g and h.
3. the *injectivity radius* $\operatorname{inj}(G, X)$ is the smallest stable translation length of a loxodromic element of G.

Combining the A- and ν-invariants, one proves the following statement: if $g_1, g_2 \in G$ are two loxodromic elements which do not generate an elementary subgroup, then

$$\operatorname{diam}\left(A_{g_1}^{+5\delta} \cap A_{g_2}^{+5\delta}\right) \leqslant [\nu(G, X) + 2] \max\left\{\|g_1\|, \|g_2\|\right\} + A(G, X) + 1000\delta,$$

see for instance Coulon [6, 9]. This inequality, generalizing Example 1.2 can be thought of as an analogue of Margulis' Lemma for manifolds with pinched negative curvature. It provides another example of local-to-global phenomena arising in hyperbolic spaces that we were mentioning before. The proof is similar to the one given in Example 1.2. So $A(G_k, X_k)$ and $\nu(G_k, X_k)$ can be used to bound from above the length of the pieces. Similarly $\mathrm{inj}(G_k, X_k)$ provides a bound from below on the length of the relations.

With these invariants we have moved the difficulty somewhere else. The goal is now to control the values of $A(G_k, X_k)$, $\nu(G_k, X_k)$ and $\mathrm{inj}(G_k, X_k)$ along the approximation sequence (1.1). The strategy for estimating $A(G_k, X_k)$ and $\mathrm{inj}(G_k, X_k)$ is based on the thin/thick decomposition of the space X_k. The arguments are somewhat similar to the ones we explained in Example 1.5. The control of $\nu(G_k, X_k)$ was also done in [6] in the absence of even torsion. As soon as even torsion is involved, the situation becomes more delicate. Indeed, the ν-invariant does not behave well when passing to a quotient. It results from the fact that the algebraic structure of finite subgroups of $\mathbf{B}_r(n)$ is rather intricate. Nevertheless, every such finite group F embeds in a product $\mathbf{D}_n \times \mathbf{D}_{n_2} \times \cdots \times \mathbf{D}_{n_2}$ where n_2 is the largest power of 2 dividing n. Thus they share numerous identities. For instance, Ivanov and Lysenok noticed that for every $g, h \in F$ we have

$$(h^3gh^{-3})(h^2gh^{-2})^{-1}g(hgh^{-1})^{-1} = \left[h^2, [h,g]\right] = 1.$$

This identity is not without connection with the ν-invariant as it involves conjugates of g by small powers of h. Taking these observations into consideration, we designed a stronger version of the ν-invariant which has a mixed nature: it combines geometric and algebraic data. Thanks to careful analysis of elementary subgroups we estimate this new invariant along the sequence (G_k). This gives us the required control on the small cancellation parameters, and thus on the exponent n.

Remark. To properly handle the strong version of the ν-invariant we mentioned above, we need n to be either odd or a multiple of 128. Note that $\mathbf{B}_r(kn)$ maps onto $\mathbf{B}_r(n)$, for all $k, n \in \mathbf{N}$. So we can still conclude that free Burnside groups of sufficiently large exponents (without further restriction) are infinite. A similar detour was already in action in the work of Ivanov [94] and Lysenok [109]. □

Remark. In the remainder of this memoir, we focus on odd exponents. We made this choice to simplify the exposition. However most of the results can be generalized to even exponents. □

1.2.3 Small cancellation over Burnside groups

As we have seen in the previous section, building infinite periodic groups is not a "piece of cake". Every time one wants to exhibit a periodic group with new features, one may have to go through the whole induction to define an approximation sequence (G_k) as in (1.1) and pass to the limit, taking care that in the process the additional relations will not collapse the construction. It makes the initial investment to enter this field pretty high.

With Dominik Gruber we developed a small cancellation theory that works in the Burnside variety $\mathfrak{B}_n$ [13]. It provides a versatile yet powerful tool to build examples of finitely generated infinite periodic groups with prescribed properties. It can be applied without any prior knowledge in the subject.

Let S be a finite set. In the same way that $\mathbf{F}(S)$ stands for the free group generated by S, we write $\mathbf{B}(S, n)$ for the free Burnside group of exponent n generated by S. Let R be a set of words over the alphabet $S \cup S^{-1}$. We assume that the elements of R are non-trivial and cyclically reduced.

Definition 1.13. Let $\lambda \in (0,1)$ and $p \in \mathbf{N}$. We say that R satisfies the $C'_p(\lambda)$ condition if

1. R satisfies the classical $C'(\lambda)$ condition; moreover no element of R is a proper power.
2. No element of R contains a subword of the form w^p.

This new small cancellation assumption has two parts. The first one is the usual $C'(\lambda)$ condition. Together with the assumption that no relation $r \in R$ is a proper power, it ensures that $\langle S|R\rangle$ is torsion-free. The second part requires that the words in R do not contain large powers. This essentially tells us that the elements of R are "transverse" to the Burnside relations of the form $x^n = 1$. We obtain the following Periodic Small Cancellation Theorem.

Theorem 1.14 (Coulon-Gruber [13]). *Let $p \in \mathbf{N}^*$. There exists $N_p \in \mathbf{N}$ with the following property. Let S be a finite set and R a set of words over $S \cup S^{-1}$. If R satisfies the $C'_p(1/6)$ condition, then for every odd exponent $n \geqslant N_p$ the quotient $\mathbf{B}(S,n)/\langle\langle R\rangle\rangle$ is infinite.*

Remark 1.2.3. Note that R can be infinite. If R is finite, it follows from classical small cancellation theory that the group $G = \langle S|R\rangle$ is torsion-free, hyperbolic. Hence by Theorem 1.6 its periodic quotients (of sufficiently large odd exponent) are infinite. However, even in this case, the Periodic Small Cancellation Theorem is not a consequence of Theorem 1.6. Indeed, the critical exponent N_p does not depend on the group G, which is the crux of the statement [1]. □

Strategy of the proof. The proof has two main steps. Instead of jumping directly to periodic groups, we first focus on the geometry of the *classical* $C'(1/6)$ small cancellation group $G = \langle S|R\rangle$. Denote by W the set of all elements of G represented by subwords of elements of R. Consider the Cayley graph X of G with respect to the (possibly infinite) generating set $S \cup W$. As graph, X is obtained from the usual Cayley graph by attaching to each embedded cycle c labeled by a relator in R a complete graph on the vertices of c [2]. Under the $C'(1/6)$ condition, Gruber and Sisto proved that X is 2-hyperbolic. Moreover the natural action of G on X is non-elementary unless G is virtually cyclic [87]. The second step of the proof is to "burnsidify" the group G by killing all possible n-th powers. To that end, we would like to apply Theorem 1.8 to the action of G on X. In particular, we need it to be acylindrical. This is where the fact that the relations of R do not contain large powers comes into play.

Assume that, contrary to our assumption, there exists a sequence of words (u_i) and exponents (p_i) diverging to infinity such that $u_i^{p_i}$ is a subword of some relation in R. Any subword of $u_i^{p_i}$ belongs to W. Hence if o stands for the vertex of X representing the identity, then $d(u_i^k o, o) = 1$, for every $k \in [\![0, p_i]\!]$. It follows that either u_i is elliptic for all but finitely many $i \in \mathbf{N}$, or the action of G on X cannot be acylindrical (indeed the stable translation length of u_i converges to zero). This example shows that if the relations of R contain arbitrarily large powers, it will be difficult to ensure that the action of G is acylindrical. It turns out that this is the only possible obstruction.

More precisely, using the second half of the $C'_p(\lambda)$ condition, we prove two facts: first every non-trivial element of G is loxodromic for its action on X; second the action of G on X is acylindrical. Moreover the functions $D, M \colon \mathbf{R}_+ \to \mathbf{R}_+$ controlling the acylindricity (see Definition 1.3) only depend on p. We can now apply Theorem 1.8 or more precisely its variation for odd exponents proved in [6]. Since G has no non-trivial elliptic element, the group Q_n studied in Theorem 1.8 is exactly G/G^n. As we explained in Remark 1.2.2, the critical exponent provided by Theorem 1.8 only depends on the acylindricity parameters of the action of G, hence on p. In particular, it does not depend on R, but only the $C'_p(1/6)$ condition it satisfies. ■

1. In [118] Ol'shanskiĭ explicitly writes that his critical exponent depends on the hyperbolicity constant of G.
2. This construction can be seen as a variation on the space $\bar{X}_\rho(\mathcal{Q})$ built in Section 1.1.2.

Definition 1.13 is actually a simplified version of our periodic small cancellation condition. Its complete form yields much more general results, encompassing Gromov's *graphical* small cancellation theory, as well as classical and graphical small cancellation theory over *free products*. The philosophy is the same though: we consider classical/graphical small cancellation presentations with additional restrictions on the powers that can appear in the relators. If these powers are restrained, then most of the standard conclusions of small cancellation theory hold. For example, n-periodic graphical small cancellation produces infinite n-periodic groups with prescribed subgraphs in their Cayley graphs, and n-periodic free product small cancellation produces infinite n-periodic quotients of free products of n-periodic groups in which each of the generating free factors embeds as a subgroup. Although the exact small cancellation assumption in this framework is slightly more technical to state, it can still be used without any prior knowledge on Burnside groups. Let us mention two applications.

Decision problems in Burnside varieties. An important question in group theory is to understand what properties of a group can be checked algorithmically. For example, for a group G given by a finite presentation $\langle S|R\rangle$, the word problem asks if there exists an algorithm which can decide whether or not a word in the alphabet $S \cup S^{-1}$ represents the identity element. It was proved by Novikov [115] and Boone [46] that finitely presented groups with unsolvable word problem exist.

A property $\mathcal{P}$ of groups is *Markov*, if there exist a finitely presented group G_+ with $\mathcal{P}$ and a finitely presented "poison group" G_- such that every group containing G_- cannot have $\mathcal{P}$. Example of Markov properties are: being trivial, abelian, nilpotent, solvable, amenable, free, etc. Building on Novikov-Boone result, Adian [26] and Rabin [135] showed the following fact: given a Markov property $\mathcal{P}$, there is no algorithm that takes a finite presentation and decides whether or not the corresponding group has $\mathcal{P}$. Roughly speaking, this means that most of the non-trivial decision problems one can think of are unsolvable in the class of all finitely presented groups. However if we restrict our attention to a smaller class (nilpotent groups, hyperbolic groups, etc), then many decision problems become solvable. It is therefore natural to ask what decision problems can be solved in the Burnside variety $\mathfrak{B}_n$. We proved the exact analogue of the Adian-Rabin Theorem in $\mathfrak{B}_n$.

Remark. It is still unknown whether there exists an infinite, *finitely presented*, periodic group. To state our result we use instead a notion of relative presentation. We say that a group $G \in \mathfrak{B}_n$ is *finitely presented relative to* $\mathfrak{B}_n$ if G is isomorphic to the quotient of a finitely generated free Burnside group $\mathbf{B}(S,n)$ by the normal closure of a *finite* subset R of $\mathbf{B}(S,n)$. Equivalently, G is the n-periodic quotient of the finitely presented group $\langle S|R\rangle$. In this situation, we refer to $\langle S|R\rangle$ as a *finite presentation of* G *relative to* $\mathfrak{B}_n$. □

Theorem 1.15 (Coulon-Gruber [13]). *There exists a critical exponent $N \in \mathbf{N}$ with the following property. Let $n \geqslant N$ be an odd integer that is not prime. Let $\mathcal{P}$ be a property of groups. Assume that there exist $G_+, G_- \in \mathfrak{B}_n$ which are finitely presented relative to $\mathfrak{B}_n$ and such that*

1. *the group G_+ has $\mathcal{P}$,*
2. *any n-periodic group containing G_- as a subgroup does not have $\mathcal{P}$.*

Then there is no algorithm that takes as input a finite presentation relative to $\mathfrak{B}_n$ and determines whether the corresponding periodic group has $\mathcal{P}$ or not.

Sketch of proof. Let H be a finitely presented group with unsolvable word problem. The standard proof of the Adian-Rabin Theorem uses a construction based on HNN extensions to show that deciding whether a group has $\mathcal{P}$ is equivalent to solving the word problem in H. Unfortunately, we do not have at our disposal an efficient analogue of HNN extensions in $\mathfrak{B}_n$. Instead we use small cancellation theory.

According to Kharlampovich, if n is a sufficiently large exponent that is not prime, then there exists a group $H \in \mathfrak{B}_n$ which is finitely presented relative to $\mathfrak{B}_n$ with unsolvable word problem [100]. Consider the following free product in $\mathfrak{B}_n$,

$$L = H *^n G_- *^n G_+ *^n \mathbf{B}_2(n)$$

and write a_1, a_2 for the generators of the factor $\mathbf{B}_2(n)$. In addition, we choose a finite generating set S for the factor $H *^n G_- *^n \mathbf{B}_2(n)$ (note that G_+ has been removed). Let w be a word over the generating set of H and $g \in H$ the elements it represents. Let $g_i = [a_i, g]$. We associate to w the group $L(w)$ obtained as the quotient of L by the finite set of relations

$$\{s = w_s(g_1, g_2) \; : \; s \in S\}, \tag{1.2}$$

where w_s is some "complicated" word in g_1 and g_2. If $g = 1$, then $L(w) = G_+$, hence $L(w)$ has $\mathcal{P}$. On the other hand, one can choose the words w_s so that if $g \neq 1$, the additional relations (1.2) satisfy a periodic small cancellation condition over free product. As a consequence, the factor G_- embeds in $L(w)$ hence $L(w)$ has not $\mathcal{P}$. It follows that deciding if a periodic group has $\mathcal{P}$ is equivalent to solving the word problem in H. Hence it is undecidable. ∎

Periodic monster groups. Using a graphical version of small cancellation theory, Gromov built finitely generated groups that coarsely contain expander graphs in their Cayley graphs [85], see also [32, 122]. As a consequence, they do not coarsely embed into Hilbert spaces, and therefore provide counter-examples to several analytic/K-theoretic conjectures, e.g. the Baum-Connes conjecture with coefficients [92]. Gromov's groups and related constructions are currently the only source of examples with this property. These constructions necessarily produce groups with infinite order elements though. Using our small cancellation theory over Burnside groups, we showed that such Gromov's monsters also exist in Burnside varieties.

Theorem 1.16 (Coulon-Gruber [13]). *There is $N \in \mathbf{N}$ such that for every odd exponent $n \geqslant N$, there exists a finitely generated group $G \in \mathfrak{B}_n$ whose Cayley graph contains an embedded (and, moreover, coarsely embedded) expander graph. In particular, G does not coarsely embed in a Hilbert space, does not have Yu's property (A), and does not satisfy the Baum-Connes conjecture with coefficients.*

Chapter 2

Automorphism groups

2.1 The growth dichotomy for automorphisms

Let G be group. The main goal of this chapter is to investigate its outer automorphism group $\mathrm{Out}(G)$ through its action on the set of conjugacy classes of G. To that end, we rely on geometric data. Assume that G acts by isometries on a metric space X. Recall that the *translation length* of an element $g \in G$ acting on X is defined as

$$\|g\|_X = \inf_{x \in X} d\,(gx, x)\,.$$

It is invariant under conjugation. Fix $\Phi \in \mathrm{Out}(G)$ and $g \in G$. We are interested in the asymptotic behavior of the map

$$k \mapsto \left\|\Phi^k(g)\right\|_X\,,$$

(we make here an abuse of notation: given $\Psi \in \mathrm{Out}(G)$, we write $\Psi(g)$ for the conjugacy class of $\psi(g)$, where $\psi \in \mathrm{Aut}(G)$ is an automorphism representing Ψ). More precisely, we would like to understand this behavior up to the equivalence defined below.

Definition 2.1 (Comparing asymptotic behaviors)**.** Given two functions $f_1, f_2 \colon \mathbf{N} \to \mathbf{R}$, we say that f_1 *grows at most like* f_2 (or f_2 *grows at least like* f_1) and write $f_1 \prec f_2$, if there exists $C \in \mathbf{R}_+^*$ such that for every $k \in \mathbf{N}$,

$$f_1(k) \leqslant C f_2(k) + C.$$

We say that f_1 *grows like* f_2 (or f_1 and f_2 are *equivalent*) and we write $f_1 \asymp f_2$, if $f_1 \prec f_2$ and $f_2 \prec f_1$. If there is no ambiguity, we often write $f_1(k) \asymp f_2(k)$ instead of $f_1 \asymp f_2$.

A map $f \colon \mathbf{N} \to \mathbf{R}$ *grows polynomially* (respectively *exponentially*) if it grows like $k \to k^d$, for some $d \in \mathbf{N}$, (respectively like $k \to \lambda^k$, for some $\lambda > 1$).

The *exponential growth rate* of Φ on an element $g \in G$ is defined by

$$\Lambda_X(\Phi, g) = \limsup_{k \to \infty} \sqrt[k]{\|\Phi^k(g)\|_X}.$$

The *spectrum* of Φ is the set

$$\mathrm{Spec}_X\,(\Phi) = \{\Lambda_X(\Phi, g) \;:\; g \in G\}\,.$$

If h is a homeomorphism of a compact manifold M, it induces an automorphism Φ of its fundamental group. In this context, the supremum of $\mathrm{Spec}_X(\Phi)$ relates to the growth rate of Φ introduced by Bowen to bound from below the entropy of h [48].

Remark. Assume that $f\colon X_1 \to X_2$ is a G-equivariant quasi-isometry between two metric spaces. There exists $C \in \mathbf{R}_+^*$ such that for every $\Phi \in \mathrm{Out}(G)$ and $g \in G$, we have

$$\frac{1}{C}\left\|\Phi(g)\right\|_{X_1} - C \leqslant \left\|\Phi(g)\right\|_{X_2} \leqslant C\left\|\Phi(g)\right\|_{X_1} + C.$$

In particular, the asymptotic behavior of $k \mapsto \|\Phi^k(g)\|_{X_i}$ in X_1 and X_2 are equivalent. Similarly,

$$\Lambda_{X_1}(\Phi, g) = \Lambda_{X_2}(\Phi, g).$$

In the remainder of this chapter, we exclusively consider proper and co-compact actions. Hence we will drop the space X from all the notations. □

Example 2.1. Let us review some groups, for which the spectrum of their automorphisms is well understood.

1. Assume that $G = \mathbf{Z}^r$ is a free abelian group. Then $\mathrm{Out}(G) = \mathrm{GL}(r, \mathbf{Z})$. Let $\Phi \in \mathrm{Out}(G)$ be represented by an invertible $r \times r$-matrix A. Our definition of spectrum relates to the one from linear algebra:

   $$\mathrm{Spec}\,(\Phi) \subset \{|\alpha| \ : \ \alpha \text{ eigenvalue of } A\} \cap [1, \infty).$$

   In particular, it is a finite set of algebraic integers. Using the Jordan normal form we observe that for every $v \in \mathbf{Z}^r$, we have

   $$\left\|A^k v\right\| \asymp k^d \lambda^k,$$

   for some $d \in [\![0, r-1]\!]$ and $\lambda \in \mathrm{Spec}(\Phi)$.
2. Let G be the fundamental group of a closed surface Σ. The modular group $\mathrm{Mod}(\Sigma)$ of Σ is a subgroup of index 2 of $\mathrm{Out}(G)$ (we required indeed in the previous chapter that mapping classes are orientation preserving). Let $\Phi \in \mathrm{Out}(G)$. According to the Nielsen-Thurston classification, up to replacing Φ by a power of Φ, there exists a collection of non-trivial disjoint simple closed curves $c_1, \dots, c_n$ on Σ which are fixed by some homeomorphism f representing Φ. Moreover the restriction of f to any connected component of $\Sigma \setminus (c_1 \cup \dots \cup c_n)$ is either a Dehn twist or a pseudo-Anosov. It follows from there that $\mathrm{Spec}(\Phi)$ is a finite set of algebraic integers. Its elements distinct from 1 correspond to the stretching factors of the pseudo-Anosov "parts" of f. Moreover, for every $g \in G$,

   $$\text{either} \quad \left\|\Phi^k(g)\right\| \asymp k, \quad \text{or} \quad \left\|\Phi^k(g)\right\| \asymp \lambda^k \quad \text{for some } \lambda \in \mathrm{Spec}\,(\Phi)\,,$$

   (λ can equal 1, for instance if g is Φ-periodic).
3. Assume that $G = \mathbf{F}_r$ is a free group. Let $\Phi \in \mathrm{Out}(G)$. The theory of (relative) train-tracks developed by Bestvina, Feighn and Haendel provides an analogue of the Jordan decomposition in the context of non-abelian free groups [44, 42]. It shows that $\mathrm{Spec}(\Phi)$ is a finite set of algebraic numbers. Elaborating on the theory of train-tracks, Levitt proved that for every $g \in G$,

   $$\left\|\Phi^k(g)\right\| \asymp k^d \lambda^k,$$

   for some $d \in [\![0, r-1]\!]$ and $\lambda \in \mathrm{Spec}(\Phi)$ [105]. □

These examples share common properties that motivate the next definition.

Definition 2.2 (Growth dichotomy)**.** Let G be a finitely generated group. An automorphism $\Phi \in \mathrm{Out}(G)$ satisfies the *growth dichotomy* if for every $g \in G$, the map $k \mapsto \|\Phi^k(g)\|$ grows either polynomially or at least exponentially. The group G satisfies the *growth dichotomy* if all its outer automorphisms do.

It follows from the previous discussion that free groups, free abelian groups and surface groups satisfy the growth dichotomy. This alternative has deep consequences, not only for $\mathrm{Out}(G)$ but also for the underlying group G. It affects for example the geometric properties of the mapping torus $M_\Phi = G \rtimes_\Phi \mathbf{Z}$ of Φ. As explained by Thurston in the "baby case" of $\Phi \in \mathrm{Out}(\mathbf{Z}^2)$, exponential growth of Φ gives rise to Sol geometry for M_Φ, whereas polynomial growth gives rise to Nil geometry [144]. See Appendix A for illustrations of these geometries.

We are interested in the following questions. What other groups do satisfy the growth dichotomy? For such groups, can we describe the spectrum of their automorphisms? What information can we extract from the growth dichotomy?

2.2 Examples and counter-examples

2.2.1 The case of hyperbolic groups

As we noticed in Example 2.1, free groups and surface groups satisfy the growth dichotomy. The next natural step is to investigate the class of hyperbolic groups. With Camille Horbez, Gilbert Levitt and Arnaud Hilion we proved the following statement.

Theorem 2.3 (Coulon-Horbez-Levitt-Hillion [17])**.** *Let G be a torsion-free hyperbolic group. Let $\Phi \in \mathrm{Out}(G)$.*

1. *The spectrum of Φ is finite.*
2. $\Lambda(\Phi, g) = \lim_{k \to \infty} \sqrt[k]{\|\Phi^k(g)\|}$*, for every $g \in G$.*
3. *For every $g \in G$, the map $k \mapsto \|\Phi^k(g)\|$ grows either polynomially or at least exponentially.*

In particular, G satisfies the growth dichotomy.

Unlike for surface groups or free groups, we are not able (yet) to provide a precise asymptotic of the form $\|\Phi^k(g)\| \asymp k^d \lambda^k$ for some $d \in \mathbf{N}$ and $\lambda \in \mathrm{Spec}(\Phi)$. Nevertheless (2) is a first step in this direction: the important fact here is the *existence* of the limit (recall that the growth rate $\Lambda(\Phi, g)$ is defined with a limit superior). The statement generalizes to toral relatively hyperbolic groups, that is torsion-free groups which are hyperbolic relative to a collection of free abelian groups.

Strategy of the proof. A standard approach to study $\mathrm{Out}(G)$ is to consider a deformation space of G: one builds a set $\mathcal{D}$ that consists of "all metric structures" of a certain kind on G and investigates the action of $\mathrm{Out}(G)$ on $\mathcal{D}$. If G is a surface group (respectively a free group) a possible deformation space is the Teichmüller space (respectively the Culler-Vogtmann outer space). Other examples are JSJ-deformation spaces, character varieties, etc.

Deformation spaces. Consider the action of G by isometries on a metric space X, which we view as a morphism $\rho\colon G \to \operatorname{Isom}(X)$. Given $\lambda \in \mathbf{R}_+^*$, we write $\lambda\rho$ for the action of G on the rescaled space λX, i.e. whose metric is given by $d_{\lambda X}(x, x') = \lambda d_X(x, x')$, for all $x, x' \in X$. Two actions $\rho_1\colon G \to \operatorname{Isom}(X_1)$ and $\rho_2\colon G \to \operatorname{Isom}(X_2)$ are called *isometric* if there exists a G-equivariant isometry from X_1 onto X_2. Similarly, ρ_1 and ρ_2 are *homothetic* if there is $\lambda \in \mathbf{R}_+^*$ such that ρ_1 and $\lambda\rho_2$ are isometric. Note that $\operatorname{Aut}(G)$ acts on the left on the set of isometric actions by pre-composition: for every $\phi \in \operatorname{Aut}(G)$ and $\rho\colon G \to \operatorname{Isom}(X)$, we write $\phi\rho$ for $\rho \circ \phi^{-1}\colon G \to \operatorname{Isom}(X)$. If $\phi \in \operatorname{Aut}(G)$ is an inner automorphism, then $\phi\rho$ and ρ are isometric.

We now fix once and for all an action $\rho_0\colon G \to \operatorname{Isom}(X_0)$ of G on a geodesic metric space X_0. We then form the following deformation space

$$\mathcal{D} = \left\{ \lambda\phi\rho_0 \ : \ \lambda \in \mathbf{R}_+^*, \phi \in \operatorname{Aut}(G) \right\} / \sim$$

where $\sim$ is the equivalence relation that identifies two isometric actions. We equip $\mathcal{D}$ with the equivariant Gromov–Hausdorff topology. It follows from the construction that $\operatorname{Out}(G)$ acts on $\mathcal{D}$. Moreover, $\mathbf{R}_+^*$ acts on $\mathcal{D}$ by homotheties. Those two actions commute. The space $\mathcal{D}$ also comes with an $\operatorname{Out}(G)$-invariant asymmetric pseudo-distance called the *Lipschitz metric*: if $\rho_1\colon G \to \operatorname{Isom}(X_1)$ and $\rho_2\colon G \to \operatorname{Isom}(X_2)$ are two points in $\mathcal{D}$, we define $\operatorname{Lip}(\rho_1, \rho_2)$ to be the best Lipschitz constant of a G-equivariant Lipschitz map from X_1 to X_2; then we let

$$d_{\mathrm{Lip}}(\rho_1, \rho_2) = \log \operatorname{Lip}(\rho_1, \rho_2).$$

The *projectivized deformation space* is defined by $\mathbb{P}\mathcal{D} = \mathcal{D}/\mathbf{R}_+^*$. It is endowed with the quotient topology.

Assume now that the space X_0 we started with is a Cayley graph of G. A construction of Bestvina-Paulin shows that $\mathbb{P}\mathcal{D}$ admits a compactification that we denote by $\overline{\mathbb{P}\mathcal{D}} = \mathbb{P}\mathcal{D} \cup \partial\mathbb{P}D$ [41, 129]. The points in its boundary can be described as minimal non-trivial isometric actions of G on $\mathbf{R}$-trees (up to G-equivariant homotheties).

A probabilistic approach. Consider an element $\Phi \in \operatorname{Out}(G)$. We want to investigate the properties of Φ by studying the dynamics of the group $\langle\Phi\rangle$ on $\mathcal{D}$ and its projectivization $\mathbb{P}\mathcal{D}$. The *drift* of Φ is the stable translation length of Φ^{-1} acting on $\mathcal{D}$, i.e.

$$\mu = \lim_{k\to\infty} \frac{1}{k} d_{\mathrm{Lip}}\left(\Phi^{-k}\rho_0, \rho_0\right).$$

It provides an upper bound on the spectrum of Φ. Indeed, unwrapping all the definitions, we get that $\Lambda(\Phi, g) \leqslant e^{\mu}$, for every $g \in G$.

Assume that the orbit of Φ in $\mathcal{D}$ is not bounded. We would like to consider the limit of the sequence $(\Phi^{-k}\rho_0)$ in $\overline{\mathbb{P}\mathcal{D}}$. However there is a difficulty here: it may very well have several accumulation points. This phenomenon is well known, when G is a free group for instance. Consider indeed an automorphism Φ of $\mathbf{F}_2 * \mathbf{F}_2$ permuting the free factors and whose restriction to each factor is given by a fully irreducible automorphism. Then $(\Phi^{-k}\rho_0)$ has several accumulation points in the compactification of the Culler-Vogtmann outer space. There is no reason to distinguish one of them. Passing to a subsequence would prevent us to get a good understanding of the full sequence (Φ^k). For instance, it could become an obstruction to prove that the limit in (2) exists.

This is the place where our proof shifts to a *probabilistic* approach. Using a method developed by Karlsson and Ledrappier [98], we construct a Φ-invariant probability measure ν on $\partial\mathbb{P}\mathcal{D}$ whose support is contained in the set of accumulation points of $(\Phi^{-k}\rho_0)$. Roughly speaking, it gives us a way to consider these accumulation points all at once. It also allows us to quantify how fast $(\Phi^{-k}\rho_0)$ approaches to infinity. By essence, Karlsson-Ledrappier's result tells us that for a generic action $\rho_\infty\colon G \to \mathrm{Isom}(T)$ on a tree T in the support of the measure ν, we have

$$\lim_{k\to\infty} \frac{1}{k} \log \mathrm{Lip}(\Phi^{-k}\rho_0, \rho_\infty) = \mu.$$

Unwrapping again all the definitions, we get that for every $g \in G$ acting *loxodromically* on T,

$$\liminf_{k\ \to\infty} \sqrt[k]{\|\Phi^k(g)\|} \geqslant e^{\mu}.$$

In particular, for those elements, the limit in (2) exists and equals e^μ.

We are now left studying subgroups of G which are elliptic for the action on T. The automorphism Φ may not fix T (seen as a point of $\partial\mathbb{P}\mathcal{D}$), however it acts on T as a bi-Lipschitz homeomorphism. Hence every point stabilizer in T is Φ-invariant. We associate to each hyperbolic group a complexity (in terms of the maximal cyclic splitting of G) so that if A is a point stabilizer in T, then its complexity is smaller than the one of G. Thus we can proceed by induction on the complexity and prove (2) for every element $g \in G$. Since the induction process stops after finitely many steps, we also get that $\mathrm{Spec}(\Phi)$ is finite.

Polynomial growth. Let us conclude with a few words about (3). According to the previous discussion, if the drift μ of $(\Phi^{-k}\rho_0)$ is positive, then for every element $g \in G$ acting loxodromically on the generic limit tree T, the map $k \to \|\Phi^k(g)\|$ grows at least exponentially. In view of the induction sketched above, we can assume that $\mu = 1$, that is for every $g \in G$, the map $k \mapsto \|\Phi^k(g)\|$ grows sub-exponentially. If G is one-ended, then (up to replacing Φ by a power) Φ preserves the JSJ-decomposition of G, see for instance Guirardel-Levitt [88]. Otherwise, it follows from the theory of train-tracks over free products that Φ preserves a non-trivial free splitting of G [106]. In both cases, Φ preserves a non-trivial decomposition over cyclic subgroups, that we denote by S. Moreover the restriction of Φ to the factors of S still grows sub-exponentially. This allows again to run a proof by induction. Suppose indeed that we already know that Φ restricted to any factor of S grows polynomially. Using Bass-Serre theory we can write any element $g \in G$ using a normal form that involves elements in the factor groups of S and stable letters. We study "by hand" the behavior of these normal forms under the iterations of Φ, and prove that for every $g \in G$, the map $k \to \|\Phi^k(g)\|$ grows polynomially. This part is slightly technical. Indeed, if some edge stabilizer of S is non trivial, we have to take care of simplifications in the normal form that can occur in the edge groups. ■

Remark 2.2.1. As in Example 2.1, $\mathrm{Spec}(\Phi)$ is actually made of algebraic integers. This requires other ingredients though: we prove this fact by comparing the deformation space $\mathcal{D}$ with the JSJ-decomposition over cyclic groups of G. □

2.2.2 Exotic automorphisms

We just added hyperbolic groups to the list of groups satisfying the growth dichotomy. Going in the opposite direction, one can ask if there exist groups which violate this dichotomy. If so, what are the possible asymptotic behaviors of their automorphisms? In order to state our next result, we first recall the definition of a length function.

Definition 2.4. A *length function* on a group H is a map $L\colon H \to \mathbf{N}$ with the following properties

1. $L(h) = 0$ if and only if $h = 1$;
2. $L(h) = L(h^{-1})$, for every $h \in H$;
3. $L(h_1 h_2) \leqslant L(h_1) + L(h_2)$, for every $h_1, h_2 \in H$;
4. the map $\mathbf{R}_+ \to \mathbf{N}$, sending r to $|\{h \in H \,:\, L(h) \leqslant r\}|$ grows at most exponentially.

For instance, the map $n \to |n|^\alpha$ is a length function on $\mathbf{Z}$, for every $\alpha \in (0, 1]$.

Theorem 2.5. *Let $L\colon \mathbf{Z} \to \mathbf{N}$, be a computable length function on $\mathbf{Z}$. There exist a finitely generated group G and an automorphism $\Phi \in \operatorname{Out}(G)$ with the following property. For every $g \in G \setminus \{1\}$, the map $k \mapsto \log \|\Phi^k(g)\|$ grows like L.*

Assume for instance that $L(k) = \sqrt{k}$. Let $g \in G \setminus \ \{1\}$. The statement tells us that there exist $\lambda_1, \lambda_2 \in (1, \infty)$ such that for every sufficiently large integer k, we have

$$\lambda_1^{\sqrt{k}} \leqslant \left\|\Phi^k(g)\right\| \leqslant \lambda_2^{\sqrt{k}}.$$

In particular, the map $k \to \|\Phi^k(g)\|$ is super-polynomial, but sub-exponential. It follows that Φ and thus G does not satisfy the growth dichotomy.

There are infinitely many inequivalent computable length functions. Hence Theorem 2.5 provides numerous examples of exotic automorphisms. We will explain the idea of the proof in the next section.

2.3 The Lipschitz metric

Definition. The proof of the growth dichotomy for hyperbolic groups made key use of a Lipschitz metric on the deformation space $\mathcal{D}$. This construction – directly inspired by the Thurston metric on Teichmüller space or the Lipschitz metric on Culler-Vogtmann outer space – can be generalized to any group.

Consider a finitely generated group G acting properly co-compactly on a metric space X. We endow $\operatorname{Out}(G)$ with an asymmetric left-invariant pseudo-metric defined as follows: for every $\Phi_1, \Phi_2 \in \operatorname{Out}(G)$,

$$d_{\mathrm{Lip}}(\Phi_1, \Phi_2) = \log\left(\sup_{g \in G\setminus\{1\}} \frac{\|\Phi_1^{-1}(g)\|}{\|\Phi_2^{-1}(g)\|}\right).$$

We call this metric the *Lipschitz metric* on $\operatorname{Out}(G)$. It does not really depend on the space X we started with. Indeed, one checks that if G acts properly co-compactly on another metric space X', then the Lipschitz metrics on $\operatorname{Out}(G)$ obtained from X' and X are quasi-isometric.

Remark 2.3.1. Note that if $\operatorname{Out}(G)$ is finitely generated, the Lipschitz metric d_{Lip} is a priori not quasi-isometric to the word metric d. Consider for instance the free group $G = \mathbf{F}(a, b)$. Let ϕ be the automorphism given by $a \mapsto a, b \mapsto ab$. Its outer class Φ can be interpreted as a Dehn twist on a punctured torus. It follows that $n \to d_{\mathrm{Lip}}(1, \Phi^n)$ grows logarithmically. However $\operatorname{Out}(G) = \mathrm{GL}_2(\mathbf{Z})$ is hyperbolic, thus $n \to d(1, \Phi^n)$ grows linearly. Hence d_{Lip} and d are not quasi-isometric. In general, d_{Lip} is not even a proper metric. Indeed, Minasyan built an example of a finitely generated group G with exactly two conjugacy classes and an infinite outer automorphism group [112]. □

Representation theorem. When studying automorphisms, a natural question is: what groups can be realized as outer automorphism groups? It turns out that any finitely presented group Q is isomorphic to $\mathrm{Out}(G)$ for some suitable finitely generated group G. See for instance Matumoto [111] and Bumagin-Wise [54]. We extended this result and proved that such a representation theorem is not only algebraic but also geometric.

Theorem 2.6 (Coulon [10])**.** *For every finitely presented group Q, there is a finitely generated group G such that $\mathrm{Out}(G)$ endowed with the Lipschitz metric is isomorphic and* quasi-isometric *to Q.*

Strategy of the proof. Let Q be a finitely presented group. According to the Rips construction there exists a short exact sequence

$$1 \to G \to H \to Q \to 1,$$

where H is a hyperbolic group and G is a *finitely generated* normal subgroup of H [136]. The action by conjugation of H on G defines a map $H \to \mathrm{Aut}(G)$ which induces a homomorphism $\chi\colon Q \to \mathrm{Out}(G)$. The group H is obtained by mean of small cancellation theory. The construction is therefore very flexible. See for instance [156, 121, 39]. For instance, one can require that the centralizer of G in H is trivial, which means that χ is one-to-one. Bumagin and Wise tweaked that construction so that χ is actually an isomorphism [54].

Since Q is finitely generated, the word metric always "dominates" any other pseudo-metric on Q. In our setting, it implies that the map χ from (Q, d) to $(\mathrm{Out}(G), d_{\mathrm{Lip}})$ is Lipschitz. In order to complete the proof, we need to establish a lower bound for the Lipschitz metric. This is the place where the hyperbolicity of H jumps in.

We choose generating sets for the groups so that the Cayley graph of G embeds in the Cayley graph X of H. Fix $q \in Q$ and write $\Phi = \chi(q)$ for its image in $\mathrm{Out}(G)$. Choose an element $g \in G$ that is loxodromic in H. The conjugacy class $\Phi(g)$ can be represented by a path γ in the Cayley graph of G whose length is $\|\Phi(g)\|$. By construction, there exists a pre-image $h \in H$ of q such that $h\gamma$ is a path joining h to gh. If h is chosen to be minimal, the geodesic from h to gh actually passes close to 1. On the other hand, $h\gamma$ lies in the coset hG. Since the projection $H \twoheadrightarrow Q$ is 1-Lipschitz, $h\gamma$ avoids a ball of radius $r = d(1, q)$ centered at 1. Hyperbolic spaces have exponential divergence, see for instance Bridson-Haefliger [51]. Hence there exists $\lambda > 1$, that does not depend on q or g, such that

$$\|\Phi(g)\| \geqslant \mathrm{Length}(\gamma) \geqslant \mathrm{Length}(h\gamma) \geqslant \lambda^{d(1,q)}.$$

Taking the logarithm, we get

$$d_{\mathrm{Lip}}(1, \chi(q)) \geqslant (\log \lambda) d\,(1, q) - \log \|g\|\,.$$

Recall that the Lipschitz metric is left-invariant. It follows from the previous inequality that χ is a quasi-isometric embedding. Since χ is also surjective, it is a quasi-isometry. ■

Remark 2.3.2. If $g \in G$ is a loxodromic element, the exact same proof shows that $k \mapsto \log \|\Phi^k(g)\|$ grows like $k \mapsto d(1, q^k)$. This can be used to produce exotic automorphisms as in Theorem 2.5. Indeed, if $L\colon \mathbf{Z} \to \mathbf{N}$ is a computable length function, Ol'shanskiĭ proved that there exist a finitely generated group Q and an element $q_0 \in Q$ such that $k \mapsto d(1, q_0^k)$ grows like L [120]. We then use Q as the input of the Rips construction. For this application, we use a version where H is torsion-free so that every non-trivial element of G is loxodromic. It follows from the previous discussion that $\chi(q_0)$ grows (after taking the logarithm) like L on any non-trivial element of G. □

2.4 Application of the growth dichotomy

We now discuss a rather surprising application of the growth dichotomy to the study of Burnside groups. Let $r, n \in \mathbf{N} \setminus \{0\}$. Recall that $\mathbf{F}_r^n$ stands for the subgroup of $\mathbf{F}_r$ generated by the n-th power of all its elements, while $\mathbf{B}_r(n) = \mathbf{F}_r / \mathbf{F}_r^n$ is the free Burnside group of rank r and exponent n. Since $\mathbf{F}_r^n$ is a characteristic subgroup, the projection $\mathbf{F}_r \twoheadrightarrow \mathbf{B}_r(n)$ induces a homomorphism

$$\mathrm{Out}\,(\mathbf{F}_r) \to \mathrm{Out}\,(\mathbf{B}_r(n))\,.$$

This map is not one-to-one though. Indeed, let $a_1, \dots, a_r$ be a free basis of $\mathbf{F}_r$. The Dehn twist $\phi \in \mathrm{Aut}(\mathbf{F}_r)$ fixing $a_2, \dots, a_r$ and sending a_1 to $a_1 a_2^n$ induces a trivial automorphism of $\mathbf{B}_r(n)$. This raises the problem of describing the kernel of the map $\mathrm{Out}(\mathbf{F}_r) \to \mathrm{Out}(\mathbf{B}_r(n))$.

With Arnaud Hilion we adopted an asymptotic point of view on this question. Our goal was to characterize the automorphisms of $\mathbf{F}_r$ that "survive" as automorphisms of $\mathbf{B}_r(n)$, provided n is *sufficiently large*. Recall that $\mathbf{F}_r$ satisfies the growth dichotomy. That is, given any $\Phi \in \mathrm{Out}(\mathbf{F}_r)$ and $g \in \mathbf{F}_r$, the map $k \mapsto \|\Phi^k(g)\|$ grows either polynomially or at least exponentially.

Theorem 2.7 (Coulon-Hilion [16]). *Let $\Phi \in \mathrm{Out}(\mathbf{F}_r)$. The following statements are equivalent.*

1. *There exists $g \in \mathbf{F}_r$ such that $k \mapsto \|\Phi^k(g)\|$ grows at least exponentially.*
2. *There is a critical exponent $N \in \mathbf{N}$, such that for every odd integer $n \geqslant N$, the automorphism Φ induces an infinite order outer automorphism of $\mathbf{B}_r(n)$.*

The direction (2) ⇒ (1) is essentially algebraic. Assume that $k \mapsto \|\Phi^k(g)\|$ grows polynomially for every $g \in \mathbf{F}_r$. Then (a power of) Φ preserves a non-trivial free splitting of $\mathbf{F}_r$ [105]. Using an induction on the rank of the free group, we prove that Φ induces a finite order automorphism of $\mathbf{B}_r(n)$.

The other direction, namely (1) ⇒ (2), relies on a careful analysis of laminations on the free group. Let $\phi \in \mathrm{Aut}(\mathbf{F}_r)$ representing Φ. Note that as an automorphism of $\mathbf{B}_r(n)$, ϕ has finite order if and only if so has Φ. Thus it suffices to focus on ϕ. As the proof uses advanced train-track technology, we prefer to explain the strategy using a few examples. Consider first the case where ϕ is fully irreducible (that is no power of ϕ preserves a proper free factor up to conjugacy). An example, studied by Cherepanov, is the automorphism ϕ of $\mathbf{F}(a,b)$ defined by $\phi(a) = ab$ and $\phi(b) = a$ [58]. If we iterate ϕ on a, it leads to the following sequence:

$$\begin{array}{lcllcl}
\phi^0(a) & = & a & \phi^4(a) & = & abaababa \\
\phi^1(a) & = & ab & \phi^5(a) & = & abaababaabaab \\
\phi^2(a) & = & aba & \phi^6(a) & = & abaababaabaababaababa \\
\phi^3(a) & = & abaab & \dots & &
\end{array}$$

Observe that $k \mapsto \|\phi^k(a)\|$ is the Fibonacci sequence, hence grows exponentially. None of the words in the above sequence contains a 4-th power [97]. If we are not picky about the exact value 4, this can be understood in terms of symbolic dynamic:

The automorphism ϕ induces a substitution σ on the free monoid generated by $\{a, b\}$. When we iterate σ on b, the resulting sequence converges (for the prefix topology) to an infinite word $w_\infty \in \{a,b\}^{\mathbf{N}}$ fixed by σ. Since ϕ is fully irreducible, σ is a primitive substitution. It follows that either w_∞ is periodic (i.e. $w_\infty = wwww\dots$, for some word w), or there exists $p \in \mathbf{N}$ such that w_∞ does not contain a p-th power [113]. So, we need to rule out the first case. The dynamics of σ is unfortunately not enough to conclude. Instead, we use the fact that σ comes from an automorphism.

Indeed, if w_∞ is a periodic fixed point of σ, then one shows that there exists a subword u of w_∞ such that $\sigma(u) = u^q$ for some $q \in \mathbf{N} \setminus \{0,1\}$. Hence q is an eigenvalue of the matrix $A \in \mathrm{GL}(2, \mathbf{Z})$ induced by ϕ on the abelianization of $\mathbf{F}(a,b)$. This is impossible, since a matrix in $\mathrm{GL}(2,\mathbf{Z})$ cannot have an integer eigenvalue distinct from ± 1.

The solution to the Burnside problem given by Novikov and Adian relies on the following important fact [28].

Proposition 2.8. *Let w be a reduced word of $\mathbf{F}_r$. If w does not contain a subword of the form u^{16} then w induces a non-trivial element of $\mathbf{B}_r(n)$ for all odd exponents $n \geqslant 655$.*

It follows that the image of $(\phi^k(a))$ in $\mathbf{B}_r(n)$ is a sequence of pairwise distinct elements, hence ϕ has infinite order as an automorphism of $\mathbf{B}_r(n)$. The same strategy can be extended to any fully irreducible automorphism of $\mathbf{F}_r$.

For reducible automorphisms, there is an additional difficulty. Consider for instance the automorphism ψ of $\mathbf{F}_4 = \mathbf{F}(a,b,c,d)$ defined by

$$a \mapsto a, \quad b \mapsto ba, \quad c \mapsto cbc^{-1}d, \quad d \mapsto c^{-1}.$$

The orbits of c under ψ grows exponentially. Note that ψ leaves the factor $\langle a,b \rangle$ invariant but not $\langle c,d \rangle$. Indeed, $\psi(c)$ contains a letter b. Each time $\psi^k(c)$ contains a subword ba^m then $\psi^{k+1}(c)$ contains ba^{m+1}. Hence as k tends to infinity, $\psi^k(c)$ contains arbitrarily large powers of a. This cannot be avoided by choosing the orbit of another element. Proposition 2.8 is no more sufficient to tell us whether or not the $\psi^k(c)$'s are pairwise distinct in $\mathbf{B}_r(n)$. An first idea would be to consider the automorphism of $\mathbf{F}_4/\langle\langle a,b \rangle\rangle$ induced by ϕ. However this operation is too brutal: the resulting automorphism has finite order [1]. We need a more accurate criterion to distinguish words in $\mathbf{B}_r(n)$. This is done using elementary moves.

Let $n \in \mathbf{N}$ and $\xi \in \mathbf{R}_+$. An (n,ξ)-*elementary move* consists in replacing a reduced word of the form $pu^m s \in \mathbf{F}_r$ by the reduced representative of $pu^{m-n}s$, provided m is an integer larger than $n/2 - \xi$. The word u is called the *support* of the elementary move. By construction a move does not change the image of a word in $\mathbf{B}_r(n)$. It may increase its length though.

Theorem 2.9 (Coulon [8])**.** *There exist $N, \xi \in \mathbf{N}$ such that for all odd exponents $n \geqslant N$ the following holds. Let w and w' be two reduced words of $\mathbf{F}_r$. If w and w' define the same element of $\mathbf{B}_r(n)$ then there are two sequences of (n,ξ)-elementary moves which respectively send w and w' to the same word.*

Remark. Although Theorem 2.9 is not explicitly mentioned in their articles, it should be possible to deduce an analogue statement from the work of Adian [28] and Ol'shanskiĭ [117]. □

Thanks to this tool we can now explain how the implication (1) ⇒ (2) of Theorem 2.7 works in general. Let us go back to our example ψ. We need to understand the effect of elementary moves on the words $\psi^k(c)$. We assign different font weights to the letters: $a^{\pm 1}$, $b^{\pm 1}$ are *regular* letters whereas $\boldsymbol{c}^{\pm 1}$, $\boldsymbol{d}^{\pm 1}$ are *bold* letters. The word $\psi^k(c)$ is the concatenation of maximal bold and regular subwords. To any word w over the alphabet $\{a,b,c,d\}$ we associate its bold part $\mathrm{Bold}(w)$ obtained by removing from w all the regular letters, without performing any cancellation. We start with two observations.

1. On our specific example we could actually consider the automorphism of $\mathbf{F}_4/\langle\langle a \rangle\rangle$ induced by ψ: the orbit of (the image of) c under this automorphism does not contain large powers. Nevertheless, one can cook up more complicated examples where this is no more the case.

Bold words. Using dynamical properties of the attractive lamination associated to the automorphism ψ, we show that the support of any elementary move that can be performed on $\psi^k(c)$ only contains regular letters, provided the exponent n is sufficiently large. Indeed, even though ψ is not irreducible, the map $\sigma\colon c \mapsto \mathrm{Bold}(\psi(c)), d \mapsto \mathrm{Bold}(\psi(c))$, defines a primitive substitution on the free monoid generated by $\{c^{\pm 1}, d^{\pm 1}\}$. Elaborating on the above argument, we prove that $\mathrm{Bold}(\psi^k(c))$ does not contain any p-th power, for some $p \in \mathbf{N}$ depending only on ψ. Consider now an exponent $n > 2p + \xi$ and assume that the support u of an (n, ξ)-elementary move performed on $\psi^k(c)$ contains a bold letter. By definition, there exists $m > n/2 - \xi$ such that u^m is a subword of $\psi^k(c)$. In particular, $\mathrm{Bold}(u)^m$ is a subword of $\mathrm{Bold}(\psi^k(c))$, a contradiction since $m > p$. It follows that the support of any (n, ξ)-elementary move only contains regular letters.

Regular words. We now claim that elementary moves with regular support cannot send a maximal regular subword of $\psi^k(c)$ to the empty word. This fact is important for the following reason. Imagine that an elementary move collapses a maximal regular subword u of $\psi^k(c)$, then the bold letters separated by u could start to cancel, thus affecting the bold part of $\psi^k(c)$. It happens for instance if we remove b from the image of $\psi(c)$.

To prove this second claim we look at the regular subwords of $\psi^k(c)$. Notice that the image by ψ of a regular word is still a regular word. On the contrary, the image of a bold word may contain regular subwords. Indeed, b is a subword of $\psi(c)$. Actually the regular subwords of $\psi^k(c)$ can be sorted in two categories. The words that consist in the single letter $b^{\pm 1}$ which appears as a subword of $\psi(c^{\pm 1})$ and the ones which arise as the images by ψ of regular subwords of $\psi^{k-1}(c)$. In particular, all the maximal regular subwords of $\psi^k(c)$ belong to the orbit under ψ of $b^{\pm 1}$. Consequently, if n is sufficiently large, none of them becomes trivial in $\mathbf{B}_r(n)$. In particular, no sequence of (n, ξ)-elementary moves send a maximal regular subword of $\psi^k(c)$ to the empty word.

We can now complete the proof. Let n be a large odd integer. Assume that ψ induces an automorphism of finite order of $\mathbf{B}_r(n)$. There exists $k \in \mathbf{N} \setminus \{0, 1\}$ such that $\psi^k(c)$ and c have the same image in $\mathbf{B}_r(n)$. By Theorem 2.9, there is a sequence of (n, ξ)-elementary moves sending $\psi^k(c)$ to c (note that no move can be performed on c). According to the previous discussion, performing (n, ξ)-elementary moves on $\psi^k(c)$ does not change its bold part. Indeed, these moves only affect the regular subwords of $\psi^k(c)$. Moreover, none of them completely disappears. Consequently $\mathrm{Bold}(\psi^k(c)) = c$. This is a contradiction. The strategy for a general automorphism of $\mathbf{F}_r$ essentially follows the same steps.

Chapter 3

Growth spectra of groups

3.1 General framework

Let G be a group acting by isometries on a pointed metric space (X, o). Unless mentioned otherwise all the actions in this chapter are proper. The *growth function* $\beta\colon \mathbf{R}_+ \to \mathbf{N}$ of this action is defined by

$$\beta(r) = \left|\{g \in G \ :\ d\,(go, o) \leqslant r\}\right|.$$

Its asymptotic behavior provides a measurement of the size of G which relates to its algebraic structure. This phenomenon is illustrated by Gromov's celebrated polynomial growth theorem: a finitely generated group has polynomial growth (for the action on its Cayley graph) if and only if it is virtually nilpotent [81]. In this chapter, we are rather interested in cases where β grows at least exponentially. The *exponential growth rate* (or *critical exponent*) of the action of G quantifies the asymptotic behavior of β. It is defined by

$$h(G, X) = \limsup_{r\to\infty} \frac{1}{r} \ln \beta(r).$$

Equivalently, it is the critical exponent of the Poincaré series

$$P(s) = \sum_{g\in G} e^{-sd(go,o)}.$$

One observes that $h(G, X)$ does not depend on the choice of the base point o. On the contrary, it is sensitive to the metric space X. Nevertheless, if $f\colon X_1 \to X_2$ is a G-equivariant quasi-isometry, then $h(G, X_1) > 0$ if and only if $h(G, X_2) > 0$.

The exponential growth rate has numerous interpretations that make it a central object in the field. Assume for instance that $G \subset \mathrm{PSL}(2, \mathbf{C})$ is a non-elementary geometrically finite kleinian group, which we let act by isometries on the hyperbolic space $X = \mathbf{H}^3$. Then $h(G, X)$ is the Hausdorff dimension of the limit set $\Lambda(G)$ of G on the Riemann sphere $\mathbf{C} \cup \{\infty\}$. It also coincides with the entropy of the geodesic flow on the unit tangent bundle of $M = X/G$ [151]. This variety of viewpoints still holds for a group G acting properly on a Gromov hyperbolic space X, but the precise formulation is more technical, see for instance [128].

Remark. If there is no ambiguity, we simply write h_G instead of $h(G, X)$ and say that h_G is the exponential growth rate / critical exponent of the group G. □

One commonly associates two *growth spectra* to the action of G on X.

— The *quotient growth spectrum* is the set

$$\{h(G/N, X/N) \ : \ N \text{ normal subgroup of } G\}.$$

— The *subgroup growth spectrum* is the set

$$\{h(H, X) \ : \ H \text{ subgroup of } G\}.$$

Note that both spectra are contained in $[0\,, h_G]$. One can also vary the definition by asking that N (respectively H) runs over a particular set of subgroups. See for instance Dahmani-Futer-Wise (for quasi-convex subgroups) [66]. We would like to investigate the properties of these sets. A first problem is to describe them, in particular, analyze their extremal values. A second goal is to understand what properties on G (or its action on X) they carry.

3.2 Quotient growth spectrum

3.2.1 Upper bound of the spectrum

Let G be a group acting properly by isometries on a metric space X. Given a normal subgroup $N \subset G$, one checks that $h(G/N, X/N) \leqslant h(G, X)$, with equality whenever N is finite. This motivates the next definition.

Definition 3.1 (Growth tightness)**.** The action of G is *growth tight* if for every infinite normal subgroup $N \subset G$, we have $h(G/N, X/N) < h(G, X)$.

Growth tightness has been intensively studied in the context of negatively curved groups, see the work of Sambusetti [139], Arzhantseva-Lysenok [35] or Arzhantseva-Cashen-Tao [31]. For instance, the action of a hyperbolic group G on its Cayley graph is growth tight [35]. When the action of G on X is growth tight, the next question is to understand if there is a gap between the growth rate of G and the one of its quotients. More generally, if $\mathcal{N}$ is a class of infinite normal subgroups of G, we would like to compute the quantity

$$\sup_{N \in \mathcal{N}} h(G/N, X/N).$$

One can mention for example the following result.

Theorem 3.2 (Arzhantseva-Guba-Guyot [34])**.** *Let $\mathbf{F}_r$ be the free group of rank r acting on its Cayley graph T with respect to a free basis. Let $\mathcal{M}$ be the class of normal subgroups N of $\mathbf{F}_r$ such that $\mathbf{F}_r/N$ is amenable. Then*

$$\sup_{N \in \mathcal{M}} h(\mathbf{F}_r/N, T/N) = h(\mathbf{F}_r, T).$$

We proved an analogue theorem where we replaced amenable quotients by periodic quotients. Recall that given a group G, we write G^n for the subgroup generated by the n-th power of all its elements.

Theorem 3.3 (Coulon [3]). *Let G be a non-elementary torsion-free hyperbolic group acting on its Cayley graph X (with respect to a finite generating set). There exists $\kappa \in \mathbf{R}_+^*$ such that for every odd integer n, we have*

$$h\left(G/G^n, X/G^n\right) \geqslant h(G,X) - \frac{\kappa}{n}.$$

In particular,

$$\sup_{n \in \mathbf{N}} h\left(G/G^n, X/G^n\right) = h(G,X).$$

Strategy of proof. The idea is to estimate the growth rate of a sufficiently large subset of G that embeds into G/G^n. Let δ be the hyperbolicity constant of X. Fix a base point $o \in X$. Given $m \in \mathbf{N}$, we say that an element $g \in G$ *contains an m-power* if there exists $h \in G \setminus \{1\}$ such that

$$\operatorname{diam}\left([o, go]^{+5\delta} \cap A_h^{+5\delta}\right) \geqslant \|h^m\|,$$

(recall that A_h stands for the axis of h in X). If G is the free group, it is equivalent to say that the reduced word representing g contains a subword of the form u^m (where u is a cyclically reduced conjugate of h). Denote by L_m the set of all elements in G that do not contain an m-th power. Theorem 2.9 generalizes to hyperbolic groups, see Coulon [8]. In particular, if n is a sufficiently large odd exponent, then $L_{n/3}$ embeds in G/G^n, Since the projection $X \to X/G^n$ is 1-Lipschitz,

$$h(G/G^n, X/G^n) \geqslant h(L_{n/3}, X).$$

Hence it suffices to count the elements in $L_{n/3}$. This computation relies on basic hyperbolic geometry. However, as it is often the case with Gromov hyperbolic spaces, it involves many parameters whose relations with one another are sometimes subtle. For simplicity, we describe here the case where $G = \mathbf{F}_r$, which goes back to Adian [28]. The general proof follows the same strategy.

Let S be a free basis of $\mathbf{F}_r$ and X the Cayley graph of $\mathbf{F}_r$ with respect to S. Let $\ell \in \mathbf{N}$. Consider a reduced word w of length $\ell+1$ that we write $w = w_0 s$ where w_0 is its prefix of length ℓ and $s \in S \cup S^{-1}$. If w_0 contains an m-th power, then so does w. Conversely, if w contains an m-th, then either w_0 contains an m-th power as well, or w has the form $w = w_1 u^m$, where $w_1 \in L_m$ and u is a non-trivial word over $S \cup S^{-1}$. It follows that

$$|L_m \cap B(\ell+1)| \geqslant \lambda\, |L_m \cap B(\ell)| - \sum_{k \geqslant 1} |L_m \cap B(\ell - mk)|\, |B(k)|\,,$$

where $\lambda = 2|S| - 1 = e^{h_G}$, while $B(\ell)$ stands for the ball of radius ℓ in $\mathbf{F}_r$. The index k in the sum corresponds to the length of u in the above discussion. The cardinality of the ball $B(k)$ is exactly $D\lambda^k$, for some constant D that does not depend on k. Writing $c(\ell)$ for the cardinality of $L_m \cap B(\ell)$, we obtain the following recurrence relation

$$c(\ell+1) \geqslant \lambda c(\ell) - D \sum_{k \geqslant 1} c(\ell - mk)\lambda^k.$$

If m is sufficiently large, then the terms in the sum have a very small contribution. More precisely, a proof by induction on ℓ shows that

$$c(\ell+1) \geqslant \left(1 - \frac{1}{m}\right)\lambda c(\ell), \quad \forall \ell \in \mathbf{N}.$$

Thus the exponential growth rate of $c\colon \mathbf{N} \to \mathbf{R}_+$ is at least $h_G + \log(1 - 1/m)$, whence the result. ■

In a recent work with Markus Steenbock, we refined this strategy to give estimates of the growth of product sets in periodic groups [25].

3.2.2 Quotient growth gap

So far we focused on the largest values in the quotient growth spectrum. We complete this discussion with a few remarks regarding the other end of the spectrum.

Note that $h(G/N, X/N) = 0$ whenever N is a finite index subgroup of G. As before one can ask if there is a gap between 0 and $h(G/N, X/N)$ when N has infinite index. If $G = \mathbf{F}_r$ is the free group acting on its Cayley tree, Grigorchuk and de la Harpe proved that there exists an increasing sequence of normal subgroups N_k such that G_k/N_k is non amenable (it even contains free subgroups) and

$$\lim_{k\to\infty} h(G/N_k, X/N_k) = 0,$$

see [78]. Actually G/N_k converges in the space of marked groups to a group with sub-exponential growth, namely the Grigorchuk group. Instead of the free group, if we started with a group G without infinite amenable quotient – for instance if G has Property (T) – it would be hard to come with a similar construction. This suggests that representation theoretic rigidity can be a useful tool to exhibit groups with a growth gap. In this context the appropriate property is Property (FM) studied by Monod and Glasner [77] or de Cornulier [62]. See also Bekka and Olivier [38].

Let us recall first some vocabulary. Let $\rho\colon G \to \mathcal{U}(\mathcal{H})$ be a unitary representation into a Hilbert space $\mathcal{H}$. Let $S \subset G$ and $\varepsilon \in \mathbf{R}_+^*$. A vector $\phi \in \mathcal{H}$ is *(S,ε)-invariant* (for the representation ρ) if $\|\rho(s)\phi - \phi\| < \varepsilon\|\phi\|$, for every $s \in S$. We say that the representation *almost has invariant vectors*, if for every finite subset $S \subset G$ and $\varepsilon \in \mathbf{R}_+^*$, there exists an (S,ε)-invariant vector.

Definition 3.4. A discrete group G has Property (FM) if for every action of G on a discrete countable set Y the following holds: if the induced unitary representation $\rho\colon G \to \mathcal{U}(\ell^2(Y))$ almost admits invariant vectors, then it has a non-zero invariant vector.

For comparison, Kazhdan Property (T) asks that *every* unitary representation of G with almost invariant vectors admits a non-zero invariant vector. Hence it implies Property (FM). The converse is not true. For example, the free product of two infinite simple groups with Property (T) has Property (FM) [77] but cannot have Property (T) since it acts on the corresponding Bass-Serre tree without global fixed point. The next statement is an analogue of the existence of Kazhdan pairs. The proof works verbatim as in [37].

Lemma 3.5. *Let G be a group with Property (FM) and $S \subset G$ a finite generating set of G. There exists $\varepsilon \in \mathbf{R}_+^*$ with the following property: for every action of G on a discrete countable set Y, if the induced representation $\rho\colon G \to \mathcal{U}(\ell^2(Y))$ has an (S,ε)-invariant vector, then it has a non-zero invariant vector.*

The next statement is a quantified version of the fact that groups with sub-exponential growth are amenable. See also Gromov [83, p. 18] and Stuck [150]. It gives examples of groups with a lower gap in their quotient spectrum.

Proposition 3.6 (Quotient growth gap)**.** *Let G be a finitely generated group with Property (FM) acting properly on a metric space X. There exists $\eta \in \mathbf{R}_+^*$, such that for every normal subgroup $N \subset G$, either $h(G/N, X/N) \geqslant \eta$ or N has finite index in G.*

Proof. Let $o \in X$ be a base point. We denote by S a finite generating set of G and let

$$a = \max_{s\in S} d(o, so).$$

Let ε be the parameter given by Lemma 3.5. Let N be a normal subgroup of G. We write Y for the image of the orbit Go in X/N. It is a countable set endowed with a transitive action of G. Let $\rho\colon G \to \mathcal{U}(\ell^2(Y))$ be the corresponding unitary representation. According to our choice of ε, either ρ has a non-zero invariant vector, or ρ does not have any (S, ε)-invariant vector. If ρ has a non-zero invariant vector, then G/N is finite (a non-zero constant function cannot be square summable, unless its domain is finite). Assume now that ρ has no (S, ε)-invariant vector. For every $n \in \mathbf{N}$, we denote by $\xi_n \in \ell^2(Y)$ the characteristic function of the ball $B(n)$ of radius na in Y. There exists $s \in S$ such that $\|\rho(s)\xi_n - \xi_n\| > \varepsilon\|\xi_n\|$. Or equivalently

$$|sB(n)\Delta B(n)| > \varepsilon^2 |B(n)|.$$

Hence either $sB(n) \setminus B(n)$ or $s^{-1}B(n) \setminus B(n)$ has cardinality at least $\varepsilon^2 |B(n)|/2$. Recall that the generators move o by at most a. Thus those sets are contained in $B(n+1) \setminus B(n)$. It follows that

$$|B(n+1)| \geqslant \left(1 + \frac{\varepsilon^2}{2}\right)|B(n)|.$$

This inequality holds for every $n \in \mathbf{N}$. Since the action of G on X is proper, we get

$$h(G/N, X/N) \geqslant \frac{1}{a}\log\left(1 + \frac{\varepsilon^2}{2}\right). \qquad \blacksquare$$

3.3 Subgroup growth spectrum

3.3.1 Amenability criterion

Let G be a group acting properly by isometries on a metric space X. For every subgroup H of G, we noticed that $h(H, X) \leqslant h(G, X)$. In this section we are interested in the following problem.

Question 3.1. When do we have $h(H, X) = h(G, X)$?

Equality holds whenever H has finite index in G. In general, it turns out that this question is intimately related to the amenability of the "quotient" G/H. This relation made its apparition in the eighties in two different contexts. Brooks studied this problem for fundamental groups of hyperbolic manifolds, while Grigorchuk and Cohen independently focused on a more combinatorial approach for free groups.

Theorem 3.7 (Brooks, [52])**.** *Let $n \in \mathbf{N}$ and $X = \mathbf{H}^{n+1}$ the $(n+1)$-dimensional hyperbolic space. Let $M = X/G$ be a convex co-compact hyperbolic manifold with $h(G, X) > n/2$. For every normal subgroup $N \subset G$, the quotient G/N is amenable if and only if $h(N, X) = h(G, X)$.*

Recall that M (or G) is *convex co-compact*, if the orbit of G on the universal cover X of M is quasi-convex.

Theorem 3.8 (Grigorchuk [79], Cohen [59])**.** *Let G be a finitely generated free group and X its Cayley graph with respect to a free basis. For every normal subgroup $N \subset G$, the quotient G/N is amenable if and only if $h(N, X) = h(G, X)$.*

These statements have been generalized in many directions. We can mention the work of Sharp (convex co-compact Schottky groups in a CAT(-1) space) [148], Stadlbauer (essentially free discrete groups of isometries on $\mathbf{H}^n$) [149], Dougall-Sharp (convex-cocompact manifolds with pinched negative curvature) [73]. All these results suggest that the relation between critical exponents and amenability holds as soon as G acts on a negatively curved space. In these examples the space X is always CAT(-1) (most of the time a tree or a Riemannian manifold) while G is either free or convex co-compact. We explored Question 3.1 in several new directions [11, 12]: what can be said if

1. N is not a normal subgroup of G?
2. the space X is Gromov hyperbolic?
3. the action of G on X is proper, but not necessarily co-compact?

We obtained the following result.

Theorem 3.9 (Coulon-Dougall-Schapira-Tapie [12]). *Let G be a group acting properly by isometries on a proper hyperbolic geodesic space X. Assume that this action is strongly positively recurrent. Let H be a subgroup of G. Then $h(H,X) = h(G,X)$ if and only if H is co-amenable in G.*

We will discuss strongly positively recurrent actions in the next section. Before, let us give the definition of co-amenability.

Co-amenability. Let H be subgroup of G. The left-action of G on G/H induces a unitary representation $\rho\colon G \to \mathcal{U}(\mathcal{H})$ where $\mathcal{H} = \ell^2(G/H)$. We say that H is *co-amenable in* G if ρ almost has invariant vectors. Equivalently, H is co-amenable in G if and only if one of the following equivalent conditions holds

— The Cheeger constant of the Schreier graph of G/H vanishes.

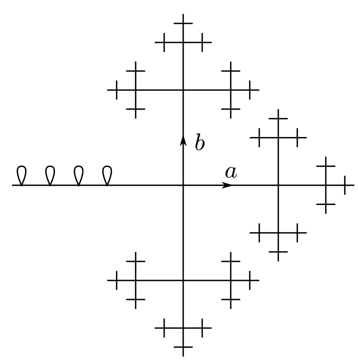
— There exists a G-invariant mean $M\colon \ell^\infty(G/H) \to \mathbf{R}$.

If H is a normal subgroup, then H is co-amenable in G if and only if G/H is amenable.

Example 3.1. Let $G = \mathbf{F}(a,b)$ be the free group generated by two elements. Consider

$$H = \left\langle a^{-n} b a^n : n \in \mathbf{N} \right\rangle$$

Roughly speaking, part of the Schreier graph of G/H is a geodesic ray, with loops attached on it (see the figure on the right). Hence its Cheeger constant is zero. Therefore H is co-amenable. $\square$



3.3.2 Strongly positively recurrent actions

Strong positive recurrence is a notion that has been defined independently by Schapira-Tapie [143] and Yang [160] (under the name *statistically convex co-compact action*). It is designed to generalize the study of convex co-compact groups.

Let G be a group acting properly by isometries on a geodesic space X. Given a compact subset $K \subset X$, we let

$$G_K = \{ g \in G \ : \ \exists x, y \in K,\ [x, gy] \cap GK \subset K \cup gK \}$$

Definition 3.10. The *entropy at infinity* of the action of G on X, that we denote by $h_\infty(G,X)$, or simply h_∞, is

$$h_\infty(G,X) = \inf_{K \subset X} h(G_K, X)$$

where K runs over all compact subsets of X. The action of G on X is *strongly positively recurrent* if $h_\infty(G,X) < h(G,X)$.

This definition has several origins. In the context of thermodynamic formalism, the entropy at infinity was used to study dynamical systems on a non-compact set. Geometrically it finds its roots in Dal'bo-Otal-Peigné [68] and can be understood as follows. Assume that X is Gromov hyperbolic and fix a base point $o \in X$. If the group G is convex co-compact, then, by definition, there exists a compact subset $K \subset X$ such that the geodesic between any two points of Go lies in GK. If G is no more convex co-compact, the set G_K collects all the "minimal" elements $g \in G$ which violate the quasi-convexity of Go. Saying that the action if strongly positively recurrent means that those pathological elements are in negligible quantity compare to the size of G.

Example 3.2. We list a few examples of strongly positively recurrent actions. More can be found in Schapira-Tapie [143] or Yang [160].

1. If G acts properly co-compactly on X, then G_K is finite, provided K is a sufficiently large compact subset. Hence $h_\infty = 0$. Thus the action of G is strongly positively recurrent as soon as G has exponential growth.
2. Let G be a group and $\mathcal{P}$ a finite collection of finitely generated subgroups of G. Assume that G acts properly by isometries on a geodesic hyperbolic space X. We say that the action of $(G,\mathcal{P})$ on X is *cusp-uniform* if there exists a G-invariant family $\mathcal{Z}$ of pairwise disjoint horoballs in X with the following properties.
   (a) The action of G on $X \setminus U$ is co-compact, where U stands for the union of all horoballs $Z \in \mathcal{Z}$.
   (b) For every $Z \in \mathcal{Z}$, the stabilizer of Z is conjugated to some $P \in \mathcal{P}$.

   Recall that the group G is *hyperbolic relative to* $\mathcal{P}$ if $(G,\mathcal{P})$ admits a cusp-uniform action on a hyperbolic space. In this context, the computation shows that
   $$h_\infty(G,X) = \max_{P\in\mathcal{P}} h(P,X).$$
   In particular, the action of G is strongly positively recurrent if and only if $h(P,X) < h(G,X)$ for every $P \in \mathcal{P}$. Hence strong positive recurrence generalizes the parabolic gap condition, introduced by Dal'bo, Otal and Peigné [68].
3. Let $M = \mathbf{H}^2/G$ be a complete hyperbolic surface with $1/2 < h(G,\mathbf{H}^2) < 1$. Denote by g_0 its Riemannian metric. For example, M can be build as a non-amenable regular cover of a compact hyperbolic surface M_0. In any pair of pants decomposition of M, choose finitely many pairs of pants $P_1,\dots,P_K$. Change the metric of M to a metric g_ε, which is equal to g_0 far from the pants P_i, and modified in the neighborhood of the P_i by shrinking the lengths of the boundary geodesics of the pants P_i to a length ε. Let G_ε be a discrete group of $\mathrm{Isom}(\mathbf{H}^2)$ such that the new hyperbolic surface (M,g_ε) is isometric to $\mathbf{H}^2/G_\varepsilon$. If ε is sufficiently small, then the action of G_ε on $\mathbf{H}^2$ is strongly positively recurrent [143]. Note that in this example G_ε is not finitely generated. □

3.3.3 Optimality of the amenability criterion

Theorem 3.9 is optimal, in the sense that no assumption can be removed without hitting a counter-example. In particular, it closes Question 3.1 for groups acting on a Gromov hyperbolic space.

Counter-example without negative curvature. Consider a finitely generated amenable group with exponential growth. For instance, a Baumslag Solitar group $\mathrm{BS}(1,n)$, a lamplighter group, or more generally, any solvable group which is not virtually nilpotent. Let X be the Cayley graph of G. Note that the action of G on X is co-compact, hence strongly positively recurrent. The trivial subgroup $H = \{1\}$ satisfies $h(H,X) < h(G,X)$ although the quotient G/H is amenable.

Following Li and Wise [107, Problem 9.4], one can ask if this problem is "fixable" by strengthening the assumption on the quotient G/H, e.g. by asking that G/H has sub-exponential growth. In general, the answer is negative. Consider indeed the lamplighter group L defined by

$$L = V \rtimes \mathbf{Z}, \quad \text{where} \quad V = \bigoplus_{n \in \mathbf{Z}} \mathbf{Z}_2.$$

The generator t of $\mathbf{Z}$ acts on V by the usual shift. Let $a = (a_n)$ be the element of V such that $a_n = 0$ if and only if $n \neq 0$. The set $\{a,t\}$ generates L. Let X be the corresponding Cayley graph of L. Parry [126] computed the growth series of L for this generating set, from which we can extract that

$$h(L,X) = \log\left(\frac{1+\sqrt{5}}{2}\right) \approx 0.481,$$

see Bucher and Talambutsa [53]. Actually Parry provides an explicit formula for the length of an element in L with respect to $\{a,t\}$ [126, Theorem 1.2]. In particular, the length $|v|$ of an element $v = (v_n)$ in V is the sum of two contributions:

1. the *length* of the shortest loop in $\mathbf{Z}$, based at the identity, that visits all indices n for which $v_n \neq 1$.
2. the *number* of indices $n \in \mathbf{Z}$ such that $v_n \neq 1$.

This can be used to compute the growth series $\zeta_V(z)$ of V for its action on X. All computations done we get

$$\zeta_V(z) = \sum_{v \in V} z^{|v|} = 1 + z + \frac{z^2(1+z)(1-z)\left(2+3z+2z^2\right)}{\left[1 - z^2(z+1)\right]^2}.$$

Hence $h(V,X)$ is (up to a logarithm) the reciprocal of largest root of $z^3 + z^2 - 1$. It approximately equals 0.281. In particular, $h(V,X) < h(L,X)$ while the quotient L/V is isomorphic to $\mathbf{Z}$.

Counter-example without strong positive recurrence. We have seen examples for which $h(H,X) < h(G,X)$ although H is co-amenable in G. We now provide examples where the other direction of our main theorem fails when we drop the strongly positively recurrent assumption.

Proposition 3.11. *Let G be a group and $\mathcal{P}$ a finite collection of residually finite subgroups of G such that G is hyperbolic relative to $\mathcal{P}$. Let X be a metric space endowed with proper cusp-uniform action of $(G,\mathcal{P})$. If the action is not strongly positively recurrent, then there exists a normal subgroup N of G such that*

1. *$h(N,X) = h(G,X)$;*
2. *G/N is non-elementary hyperbolic, hence non-amenable.*

Sketch of proof. In view of Example 3.2 (2) we can find a parabolic subgroup $P \in \mathcal{P}$, such that $h(P,X) = h(G,X)$. According to the group theoretic Dehn filling [86, 124], there exists a finite index subgroup P_0 of P such that the quotient of G by $N = \langle\langle P_0 \rangle\rangle$ is non-elementary hyperbolic. Since P_0 is a finite index subgroup of P, it has the same growth rate as P, i.e. $h(G,X)$. As N contains P_0, its growth rate is also $h(G,X)$. ∎

Let Σ be locally CAT(-1) finite volume surface with finitely many cusps. Its fundamental group G is hyperbolic relative a collection $\mathcal{P}$ of infinite cyclic subgroups. Moreover the action of G on the universal cover X of Σ is cusp-uniform. Dal'bo, Otal and Peigné built examples of such surfaces with pinched negative curvature for which $h(G, X) = h(P, X)$ for some $P \in \mathcal{P}$ [69]. Proposition 3.11 tells us that the amenability criterion (Theorem 3.9) fails in this setting.

3.3.4 Let's Twist Again

Theorem 3.9 and its variations (see Theorem 3.13) are the peak of two collaborations. The first step with Françoise Dal'bo and Andrea Sambusetti establishes Theorem 3.9 when G is a hyperbolic group endowed with the word metric or a CAT(-1) metric [11]. It has been an important source of inspiration for the second part carried with Rhiannon Dougall, Barbara Schapira and Samuel Tapie [12]. Before we explain our strategy let us comment the techniques used by other authors and their limitations.

In their proof of Theorem 3.8 Grigorchuk and Cohen run a delicate explicit computation to relate the growth rates of G and N to the spectral radius of the random walk in G/N. In this way the statement reduces to Kesten's amenability criterion [99]. This computation relies on the fact that free groups are isotropic (every generator plays the same role) and would become extremely cumbersome to run in a more general setting. Brooks' approach of Theorem 3.7 makes use of spectral theory. He shows that the amenability of G/N is equivalent to the absence of a gap in the bottom of the spectrum of the Laplace-Beltrami operator. The observation is then combined with Sullivan's formula which relates the eigenvalues of this operator to the growth rates of G and N [152]. This method is rather specific to differential geometry. Indeed, it is not clear how to define a suitable Laplace operator in the context of Gromov hyperbolic spaces. Another strategy exploits the thermodynamical formalism. The first step is to establish an analogue of Kesten's criterion that works for the group extension of an irreducible subshift (and not just a random walk) [149, 11]. Then, using geometry, one encodes the geodesic flow on X by a subshift with a suitable potential so that its entropy relates to the growth rates of G and N [73, 11]. Unfortunately in the large framework of Theorem 3.9 such a coding may not exist. For these reasons our proof introduces a new tool called twisted Patterson-Sullivan measures.

Theorem 3.9 is a combination of two results. The "easy direction" states that if H is co-amenable in G, then $h(H, X) = h(G, X)$. An elegant proof of this direction when H is normal is due to Roblin [138]. We will focus here on the other direction, that we call the "hard direction". In order to avoid complications due to hyperbolicity, we assume that the space X is CAT(-1). We denote by ∂X its visual boundary and write $\bar{X} = X \cup \partial X$ for its compactification. We fix a base point $o \in X$. We write $C(\bar{X})$ for the space of continuous functions on $\bar{X}$.

Let $h \in \mathbf{R}_+$. An *h-conformal density* is a collection $\nu = (\nu_x)_{x \in X}$ of finite measures on ∂X such that for every $x, y \in X$.

$$\frac{d\nu_x}{d\nu_y}(\xi) = e^{-h\beta_\xi(x,y)}, \quad \forall \xi \in \partial X,$$

where β_ξ stands for the Busemann function at the point $\xi \in \partial X$. Such a density is *G-invariant* if $g_*\nu_x = \nu_{gx}$, for every $g \in G$ and $x \in X$. Patterson and Sullivan proved that a G-invariant h-conformal density always exists for $h = h_G$. Let us quickly recall this construction for later comparison with its twisted version. Given $s > h_G$ and $x \in X$, one builds a measure on $\bar{X}$ as a sum of Dirac masses weighted by the Poincaré series

$$\nu_x^s = \frac{1}{P(s)} \sum_{g \in G} e^{-sd(o,go)} \mathrm{Dirac}(go).$$

(Recall that $P(s)$ stands for the Poincaré series on G.) Up to passing to a subsequence, ν_x^s converges (for the weak-$*$ topology) when s tends to h_G to a measure ν_x so that $\nu = (\nu_x)_{x \in X}$ is a G-invariant h_G-conformal density [1]. It is usually called the *Patterson-Sullivan density*.

Consider now a Hilbert space $\mathcal{H}$ endowed with a partial order $\prec$ which is compatible with the Hilbert structure, that is

1. $\phi_1 \prec \phi_2 \Longrightarrow \lambda\phi_1 + \psi \prec \lambda\phi_2 + \psi$, for every $\phi_1, \phi_2, \psi \in \mathcal{H}$ and $\lambda \in \mathbf{R}_+$:
2. for every $\phi_1, \phi_2 \in \mathcal{H}$, the set $\{\phi_1, \phi_2\}$ has a least upper bound and a greatest lower bound;
3. $(\phi_1, \phi_2) \geqslant 0$ for every $\phi_1, \phi_2 \in \mathcal{H}$, such that $0 \prec \phi_1$ and $0 \prec \phi_2$.

We call the pair $(\mathcal{H}, \prec)$ a *Hilbert lattice*, see Schaefer [142]. We write $\mathcal{H}^+$, for the *positive cone*, i.e. the set of vectors $\phi \in \mathcal{H}$ satisfying $0 \prec \phi$. A bounded operator A on $\mathcal{H}$ is *positive* if it preserves $\mathcal{H}^+$. We denote by $\mathcal{B}^+(\mathcal{H})$ the space of positive bounded operators on $\mathcal{H}$. Fix a unitary representation $\rho\colon G \to \mathcal{U}(\mathcal{H})$. We assume that ρ is *positive*, i.e. $\rho(g)$ is a positive operator for every $g \in G$.

Example 3.3. Assume that G is a acting on a set Y. Then $\mathcal{H} = \ell^2(Y)$ is a Hilbert lattice, where the order $\prec$ is defined as follows: given any two square summable functions $f_1, f_2 \colon Y \to \mathbf{R}$, we let $f_1 \prec f_2$, if $f_1(y) \leqslant f_2(y)$ for every $y \in Y$. The action of G on Y induces a positive unitary representation $\rho\colon G \to \mathcal{U}(\mathcal{H})$. □

By analogy with the standard Poincaré series, we defined a *twisted Poincaré series* [2]

$$A(s) = \sum_{g \in G} e^{-sd(o,go)} \rho(g).$$

The topology used here is the strong operator topology. Note that we are summing positive operators. Hence the convergence can be reformulated as follows: the series $A(s)$ converges if and only if there exists $M \in \mathbf{R}_+$ such that for every finite subset $S \subset G$

$$\left\| \sum_{g \in S} e^{-sd(o,go)} \rho(g) \right\| \leqslant M.$$

Using monotone convergence, we are able to associate a critical exponent to $A(s)$: there exists $h_\rho \in \mathbf{R}_+$ such that for every $s > h_\rho$ (respectively $s < h_\rho$) the series converges (respectively diverges). It follows from the triangle inequality that $h_\rho \leqslant h_G$.

Example 3.4. Let H be a subgroup of G. Consider the regular representation $\rho\colon G \to \mathcal{U}(\mathcal{H})$ where $\mathcal{H} = \ell^2(G/H)$. Denote by $\phi_0 \in \mathcal{H}$, the Dirac mass at the coset H. The computation shows that

$$\sum_{g \in H} e^{-sd(o,go)} \leqslant (A(s)\phi_0, \phi_0).$$

It follows that $h_H \leqslant h_\rho \leqslant h_G$. In particular, if $h_H = h_G$, then $h_\rho = h_G$. □

1. We purposely oversimplified the construction: if the action of G is convergent (i.e. the Poincaré series converges at $s = h_G$) one needs to adjust the weights in the Poincaré series so that the limit measure ν_x is supported on ∂X. Nevertheless this modification will not affect our arguments.
2. This idea of twisting classical objects was already present in [11] which deals with twisted transfer operators.

The previous example tells us that in order to prove the “hard” direction of Theorem 3.9 it suffices to prove the following fact: if $h_\rho = h_G$, then ρ almost has invariant vectors. This is the purpose of the next result.

Theorem 3.12. *Let G be a group acting properly by isometries on a proper geodesic hyperbolic space X. Assume that this action is strongly positively recurrent.*

For every finite subset $S \subset G$, for every $\varepsilon \in \mathbf{R}_+^*$, there exists $\eta \in \mathbf{R}_+^*$ with the following property. Let $\rho\colon G \to \mathcal{U}(\mathcal{H})$ be a unitary positive representation of G into a Hilbert lattice. If $h_\rho \geqslant h_G - \eta$, then ρ has an (S, ε)-invariant vector.

Strategy of the proof. We will focus here on a slightly weaker statement: if $h_\rho = h_G$, then ρ almost has invariant vectors. To that end, we twist the Patterson-Sullivan construction. Given $x \in X$ and $s > h_\rho$, we define a measure on $\bar{X}$ taking values in $\mathcal{B}^+(\mathcal{H})$ as follows [3]

$$a_x^s = \frac{1}{\|A(s)\|} \sum_{g \in G} e^{-sd(x,go)} \mathrm{Dirac}(go)\rho(g).$$

We would like to make these measures converge when s tends to h_ρ. The untwisted construction exploits here the fact that the space of probability measures on $\bar{X}$ is compact. Unfortunately the space of operator valued measures is not necessarily compact. To force convergence, we use ultra-limits, see for instance Druţu and Kapovich [74].

Let $\omega\colon \mathcal{P}(\mathbf{N}) \to \{0,1\}$ be a non-principal ultra-filter. Given a sequence of Hilbert spaces $(\mathcal{H}_n)$ we write $\mathcal{H}_\omega = \lim_\omega \mathcal{H}_n$ for its ω-limit. The norm on each $\mathcal{H}_n$ yields a norm on $\mathcal{H}_\omega$, which makes it a Hilbert space as well. If (B_n) is a bounded sequence of operators on the spaces $\mathcal{H}_n$, then we define a bounded operator $B_\omega = \lim_\omega B_n$ on $\mathcal{H}_\omega$ by the relation

$$B_\omega \left(\lim_\omega \phi_n \right) = \lim_\omega \left(B_n \phi_n \right).$$

For our purpose we take for $(\mathcal{H}_n)$ the constant sequence equal to $\mathcal{H}$. In particular, the representation $\rho\colon G \to U(\mathcal{H})$ yields a representation $\rho_\omega\colon G \to \mathcal{U}(\mathcal{H}_\omega)$. It is a standard fact that ρ almost has invariant vectors if and only if ρ_ω has a non-zero invariant vector. Thus our goal is to prove that ρ_ω has a non-zero invariant vector.

Fix a sequence (s_n) of real numbers converging from above to h_ρ. We define the limit measure a_x by

$$\int f da_x = \lim_\omega \int f da_x^{s_n}, \quad \forall f \in C(\bar{X}).$$

It is a measure on $\bar{X}$ taking values in $\mathcal{B}(\mathcal{H}_\omega)$.

Remark. Considering operator valued measures can be unsettling at first. In practice, we work instead with linear functionals from $C(\bar{X})$ to $\mathcal{B}(\mathcal{H}_\omega)$. Note however that the Riesz representation theorem – building a one-to-one correspondence between linear functionals and measures – is not necessarily true in this context. Nevertheless, in this section, we keep the point of view of operator valued measures as it simplifies the statements, without affecting their essence. □

We call $a = (a_x)$ a *twisted Patterson-Sullivan density.* It shares many properties with the standard Patterson-Sullivan densities:

3. As for standard Patterson-Sullivan measures, we may have to adjust the weights in the twisted Poincaré series to make sure that $A(s)$ diverges at $s = h_\rho$. Otherwise the twisted measure would not be supported on ∂X. But let us forget this detail here.

1. (*Support*) The measure a_x is supported on ∂X.
2. (*Normalization*) The operator $\displaystyle\int \mathbb{1} da_o$ has norm 1.
3. (*Twisted invariance*) $g_* a_x = \rho_\omega(g)^{-1} a_{gx}$, for every $x \in X$ and $g \in G$,
4. (*Conformality*) $\dfrac{da_x}{da_y}(\xi) = e^{-h_\rho \beta_\xi(x,y)}\mathrm{Id}$, for every $x, y \in X$.

Note in particular the twisted invariance. It will play a crucial role. Once the twisted measures are built, the rest of the proof becomes particularly simple. Given $x, y \in X$ and $r \in \mathbf{R}_+$, we denote by $\mathcal{O}_x(y,r) \subset \partial X$ the shadow of the ball $B(y,r) \subset X$ seen from x, that is the set of points $\xi \in \partial X$ so that the geodesic ray from x to ξ crosses the ball of radius r centered at y. An important fact about Patterson-Sullivan measures is the Shadow Lemma. It states that for every $g \in G$, the measure of $\mathcal{O}_o(go,r)$ is essentially

$$\nu_o\left(\mathcal{O}_o(go,r)\right) \simeq e^{-h_G d(o,go)}. \tag{3.1}$$

It twisted analogue tells us that there is a constant $C \in \mathbf{R}_+^*$ such that

$$\left\|a_o\left(\mathcal{O}_o(go,r)\right)\right\| \leqslant C e^{-h_\rho d(o,go)}, \quad \forall g \in G. \tag{3.2}$$

Hence if $h_\rho = h_G$, then ν_o "dominates" a_o on shadows. The strong positive recurrence of the action is used to prove that the measure a_x gives full mass to the *radial limit set* of G. This apparently technical result allows us to approximate the measure of any Borel set by measures of shadows. Using a Vitali argument we conclude from (3.1) and (3.2) that a_o is absolutely continuous with respect to ν_0. The corresponding "Radon-Nikodym derivative"

$$D = \frac{da_o}{d\nu_o}$$

is a function from ∂X to $\mathcal{B}(\mathcal{H}_\omega)$. The (twisted) invariance of ν and a directly implies that

$$D(g\xi) = \rho_\omega(g) D(\xi), \quad \nu_o\text{-}a.s. \tag{3.3}$$

Since the action of G is strongly positively recurrent, it is divergent (i.e. the standard Poincaré series diverges at $s = h_G$). According to the Hopf-Tsuji-Sullivan dichotomy, the diagonal action of G on $(\partial X \times \partial X, \nu_o \otimes \nu_o)$ is ergodic, see Roblin [137]. Using this fact we prove that D is essentially constant. Since the total "mass" of a_o is 1, the essential value of D is a non-zero bounded operator on $\mathcal{H}_\omega$. By (3.3) its image consists of ρ_ω-invariant vectors. Consequently ρ_ω admits a non-zero invariant vector, which was our goal. ■

3.3.5 Subgroup growth gap

Theorem 3.9 completely characterizes the subgroups H whose growth rate equals the one of G. The next question is to understand if there is a gap between $h(G,X)$ and the other possible values in the subgroup growth spectrum of G. Theorem 3.12 directly leads to the following wide generalization of Corlette's growth rigidity result [61], see also [72, 11]. It echoes the quotient growth gap stated in Proposition 3.6. We refer to Section 3.2.2 for the definition of Property (FM).

Theorem 3.13 (Coulon-Dougall-Schapira-Tapie [12])**.** *Let G be a group with a proper, strongly positively recurrent action on a proper geodesic hyperbolic space X.*

If G has Property (FM) then the following holds: there exists $\eta \in \mathbf{R}_+^*$ such that for every subgroup H of G, either $h(H,X) \leqslant h(G,X) - \eta$ or H is a finite index subgroup of G.

Chapter 4

Perspectives

As we have seen in the previous chapters, the study of periodic groups can be a starting point to explore various branches of mathematics. We expose here a few open questions on the Burnside varieties. In our exposition we mostly focus on the free Burnside group $\mathbf{B}_r(n)$, but most of the questions are relevant for periodic quotients of negatively curved groups as well.

Action on a metric space. A standard strategy in geometric group theory is to exploit the action of a group G on a suitable metric space X (typically with some kind of negative curvature) to extract information on the algebraic structure of G. When it comes to periodic groups, we face an important obstacle: one does not know any "good" action of $\mathbf{B}_r(n)$ on a metric space. For instance, if $\mathbf{B}_r(n)$ acts by isometries on a hyperbolic space, then it is necessarily elliptic or parabolic, which makes hyperbolicity useless. Nevertheless, as we saw in Chapter 1, negative curvature plays an important role in the study of periodic groups. Indeed, $\mathbf{B}_r(n)$ can be seen as the limit of a sequence of groups (G_k)

$$\begin{array}{ccccccccccccc} \mathbf{F}_r = G_0 & \twoheadrightarrow & G_1 & \twoheadrightarrow & G_2 & \twoheadrightarrow & \cdots & \twoheadrightarrow & G_k & \twoheadrightarrow & G_{k+1} & \twoheadrightarrow & \cdots \\ \circlearrowright & & \circlearrowright & & \circlearrowright & & & & \circlearrowright & & \circlearrowright & & \\ X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_k & \longrightarrow & X_{k+1} & \longrightarrow & \cdots \end{array} \tag{4.1}$$

where each G_k acts on a hyperbolic space X_k obtained by geometric small cancellation. We wonder if there is a geometry for $\mathbf{B}_r(n)$ that will "remember" this construction.

In [85] Gromov sketched a new notion of negative curvature specially designed for Burnside groups. Roughly speaking, a metric space X is *fractally hyperbolic* if there exists $\delta \in \mathbf{R}_+$ such that for every $r \in \mathbf{R}_+$ any ball of radius r in X is δr hyperbolic. The definition is relevant only for small values of δ, otherwise it is vacuous. There is here a new class of spaces that we started to explore. Consider the direct limit X_∞ of the metric spaces X_k given in (4.1). The Cayley graph Γ of $\mathbf{B}_r(n)$ embeds in X_∞. The spaces X_k are δ_k-hyperbolic, with δ_k tending to infinity. Moreover the maps $X_k \to X_{k+1}$ are injective when restricted to larger and larger balls. As a consequence, the metric of X_∞ to Γ provides an example of a fractally hyperbolic space, that we will denote Γ_∞. The difficulty though is that this space is not geodesic. Indeed, a geodesic, fractally hyperbolic space is always hyperbolic (provided δ is sufficiently small) which cannot be the case for Γ_∞. Actually one can find paths in Γ_∞ whose length is the shortest among all paths with the same endpoints, but which are arbitrarily distorted in Γ_∞.

We would like to understand the properties of fractally hyperbolic spaces and the groups acting on them. This would provide a framework to study any group obtained by iterated small cancellation, far beyond the class of periodic groups.

Leaving the world of negative curvature, it is not known if infinite periodic groups can act properly on a (proper) CAT(0) space. Recall that it is still an open question whether CAT(0) groups satisfy the Tits' alternative.

Analytic properties. It is easy to produce infinite periodic groups with Kazhdan Property (T). Consider indeed a torsion-free lattice G in $\mathrm{Sp}(r,1)$, for some integer $r \geqslant 2$. It is a hyperbolic group with Property (T). Since Property (T) is stable under taking quotients, G/G^n is an infinite periodic groups with Property (T), provided n is a sufficiently large exponent (see Theorem 1.7). More generally, Shalom asked whether every periodic group has Property (T) [147]. Using a beautiful construction he called *cubization*, Osajda proved that $\mathbf{B}_r(n)$ admits an unbounded action on a CAT(0) cube complex [123], hence answering Shalom's question negatively.

Haagerup Property (also known as a-T-menability) is a strong negation of Property (T). In view of the above discussion, it is natural to ask whether $\mathbf{B}_r(n)$ has Haagerup Property. Exhibiting a proper action of $\mathbf{B}_r(n)$ on a CAT(0) cube complex, would answer the question positively. Unfortunately Osajda's action is not proper. Together with Vincent Guirardel we made a first step in this direction [15]. In a series of prominent works [157, 158, 159] Wise developed a small cancellation theory in the context of CAT(0) cube complexes with groundbreaking applications. The most famous one being Agol's solution of the virtual Haken conjecture [30]. Adapting Wise's cubical small cancellation theory, we produced the first examples of finitely generated, non-amenable, torsion groups acting properly on an (infinite dimensional) CAT(0) cube complex. The idea is to endow each space X_k in (4.1) with a wall structure, such that the action of G_k on the dual CAT(0) cube complex C_k is proper. A careful control of this new structure is used to prove that "at the limit" we keep a proper action on a CAT(0) cube complex. Our method does not produce groups with bounded torsion yet. The difficulty, which comes from controlling the *wall pieces*, is not unrelated to the challenges that need be overcome to study periodic groups with even exponents. We hope that our understanding of even torsion will be helpful in this context.

Amenable groups are examples of groups with Haagerup property. Although Adian proved that free Burnside groups are not amenable [29], it is not known whether there exists a finitely generated, infinite, amenable, periodic group. This is a question that we would like to investigate. Note that if it exists such a group cannot be elementary amenable (otherwise it would be finite).

Another analytic property we are interested in is the *Rapid Decay Property.* A group G has the Rapid Decay Property if the set of rapidly decaying functions on G is contained in the reduced C^*-algebra of G. Rapid decay has many applications in K-theory, C^*-algebra and random walks, see Chatterji [56] for a survey. In particular, groups with rapid decay have the "tendency" to satisfy the Baum-Connes conjecture. The question whether or not periodic groups have the Rapid Decay Property was raised by Sapir [141]. For many groups with non-positive curvature (e.g. hyperbolic groups [96], infinitely presented small cancellation groups [33], groups acting on CAT(0) cube complexes [57], etc) the proof of rapid decay relies on geometric arguments. Investigating rapid decay for $\mathbf{B}_r(n)$ would be a challenge to check if we really understand its geometry.

Dynamical systems. We already mentioned that free Burnside groups are non-amenable. It suggests that they can have a very rich dynamics. For instance, given a measure μ on $\mathbf{B}_r(n)$ (supported on a finite generating set say), the Poisson-Furstenberg boundary associated to the corresponding random walk is non-trivial. It would be interesting to study such random walks and give a description of the corresponding Poisson boundary. Fractal hyperbolicity could be useful in this settings. Indeed, random walks in hyperbolic groups and their generalizations have been intensively studied. It is possible that the local hyperbolicity satisfied at each scale in Γ_∞ suffices to adapt some of the arguments developed in negative curvature.

A related problem is to compactify $\mathbf{B}_r(n)$. Indeed, very often the structure of a group is reflected in the dynamics of the action induced on its boundary. There are various classical constructions to do so (one point compactification, Floyd compactification, horocompactification, Stone-Čech compactification, etc). But the resulting boundaries are either too small or too wild. It would be interesting to have a more workable compactification for $\mathbf{B}_r(n)$, for instance one that could serve as a topological model for the Poisson boundary.

More generally, we would like to inject ideas from dynamical systems and ergodic theory in the study of the Burnside varieties. We believe that not only it will give new insights on periodic groups, but it will also provide interesting examples for people in this field.

First order logic. Model theory provides another way to approach group theory. Roughly speaking, the idea is to understand a group G via its first order theory, i.e. the set of all elementary formulas (in the language of groups) satisfied by G. For instance, given an integer $n \in \mathbf{N}$, the group G belongs to $\mathfrak{B}_n$ if and only if it satisfies the formula $\forall x,\ x^n = 1$. (Note that n is fixed here. Indeed, the language of groups does not allow to quantify on integers.) The Tarski problem has been a striking motivation in the area. It asks whether two non-abelian finitely generated free groups have the same first order theory. The question was answered positively by Sela [146] and independently by Kharlampovich and Myasnikov [101].

Together with Zlil Sela, we started to investigate periodic groups, from a logical point of view. A (long term) goal of this project is to understand the periodic analogue of the Tarski problem: given $n \in \mathbf{N}$, have two non-abelian free Burnside groups of exponent n the same first order theory. Sela's solution of the Tarski problem borrows many concepts from geometric group theory and low dimension topology. Quite surprisingly, some of these techniques can be adapted in our context. Let us give an illustration of the current status of our work. A group G is co-Hopfian if every monomorphism from G to itself is actually an isomorphism. If G is a torsion-free hyperbolic, it is known that G is co-Hopfian if and only if G is freely indecomposable. Here is the corresponding analogue in the Burnside variety.

Theorem 4.1 (Coulon-Sela)**.** *Let G be a torsion-free hyperbolic group with no root splitting. There exists $N \in \mathbf{N}$ such that for every odd integer $n \geqslant N$, the following statements are equivalent.*

1. *G is freely indecomposable;*
2. *G/G^n is freely indecomposable in $\mathfrak{B}_n$;*
3. *G/G^n is co-Hopfian.*

We say that G admits a *root splitting* if G decomposes as $G = A *_C B$ where B is isomorphic to $\mathbf{Z}$ and C is a proper finite-index subgroup of B. This assumption is crucial. Indeed, fix an integer $p \in \mathbf{N} \setminus \{0, 1\}$ and consider the group G given by

$$G = \langle a, b, c \mid c^p = [a, b] \rangle .$$

It follows from Bestvina-Feighn Combination Theorem that G is hyperbolic. One checks as well that G is freely indecomposable. Nevertheless, if n and p are co-prime, then G/G^n is isomorphic to $\mathbf{B}_2(n)$, which is neither freely indecomposable nor co-Hopfian. Theorem 4.1 generalizes to any torsion-free hyperbolic group G, but one needs to add restrictions on the allowed exponents.

The next step in this project is to develop an analogue of Makanin-Razborov diagrams to describe the set of solutions of a given equation in $\mathbf{B}_r(n)$. It turns out that the existence of roots as in our previous example is a major source of complications that need be overcome.

Automorphisms. Despite some progresses, the outer automorphism group of $\mathbf{B}_r(n)$ is still very mysterious. Recall that the projection $\mathbf{F}_r \twoheadrightarrow \mathbf{B}_r(n)$ induces a morphism $\chi\colon \operatorname{Out}(\mathbf{F}_r) \to \operatorname{Out}(\mathbf{B}_r(n))$ which is neither one-to-one nor onto. Since $\operatorname{Out}(\mathbf{F}_r)$ is the object of intensive research, it would be interesting to understand what properties of $\operatorname{Out}(\mathbf{F}_r)$ survives in $\operatorname{Out}(\mathbf{B}_r(n))$. For instance, it is known that $\operatorname{Out}(\mathbf{F}_r)$ satisfies the Tits' alternative [42, 43]. Does it also holds for $\operatorname{Out}(\mathbf{B}_r(n))$, or at least for the image of χ?

The very first task is to describe the kernel of χ. Together with Vincent Delecroix, we are in the process of solving this question when $r = 2$. Recall that $\operatorname{Out}(\mathbf{F}_2)$ can be seen as the mapping class group of the torus with one boundary component. We prove that if n is a sufficiently large odd exponent, then $\ker \chi$ is the normal subgroup generated by the n-th power of every Dehn-twist. The proof carefully analyzes powers appearing in the abstract dual lamination associated to each outer automorphism of $\mathbf{F}_2$, see for instance [63, 64, 65]. For the moment it relies on some dynamical features of substitutions which are specific to $\mathbf{F}_2$, but we hope to extend the method in higher rank.

Another problem is to understand the image of χ, or more precisely what automorphisms of $\mathbf{B}_r(n)$ are *not* in this image. A typical example is the following. Let $\{a_1, a_2, \dots, a_r\}$ be the image in $\mathbf{B}_r(n)$ of a free basis of $\mathbf{F}_r$. Consider an integer p that is co-prime with n. Let Φ_p be the automorphism $\mathbf{B}_2(n)$ given by $a_1 \mapsto a_1^p$ and $a_i \mapsto a_i$, for every $i \neq 1$. If $p \neq \pm 1 \mod n$, then Φ_p does not come from $\operatorname{Out}(\mathbf{F}_r)$. The map $\zeta\colon (\mathbf{Z}/n\mathbf{Z})^* \to \operatorname{Out}(\mathbf{B}_r(n))$ sending p to Φ_p is another homomorphism. We wonder if $\operatorname{Out}(\mathbf{B}_r(n))$ is generated by the images of χ and ζ. Recall that it is unknown whether $\operatorname{Out}(\mathbf{B}_r(n))$ is finitely generated.

In Chapter 2, we defined and investigated a growth dichotomy for automorphisms (see Definition 2.2). After manipulating numerous examples we came with the idea that a stronger dichotomy might hold in $\operatorname{Out}(\mathbf{B}_r(n))$: given an outer automorphism Φ and $g \in \mathbf{B}_r(n)$, the map $k \mapsto \Phi^k(g)$ either is periodic or its length grows at least exponentially. A first stop is to understand if this property holds for automorphisms in the image of χ. It seems that the tools we develop in order to study the kernel of χ, would give a positive answer when $r = 2$.

In order to study the outer automorphism group of a group G, one often uses the action of $\operatorname{Out}(G)$ on a deformation space of G. It would be very helpful to have a deformation space for $\mathbf{B}_r(n)$. In our approach of the Burnside problem, we use geometric small cancellation theory to study the periodic quotients of a negatively curved G. The input of this construction is not just a group G, but an action of G on a hyperbolic space. Recall that a point in the Culler-Vogtmann outer space CV_r is a free minimal action of $\mathbf{F}_r$ on a simplicial tree. Hence for each point in CV_r we could run the procedure summarized in (4.1) and produce a fractally hyperbolic space. The collection of all spaces obtained in this way could be a prototype for a deformation space of $\mathbf{B}_r(n)$. Although by construction it should come with an action of the image of χ, it is not clear whether it would be invariant under $\operatorname{Out}(\mathbf{B}_r(n))$. There are still numerous questions to investigate in this direction.

Appendix A

Ray-marching Thurston's geometries

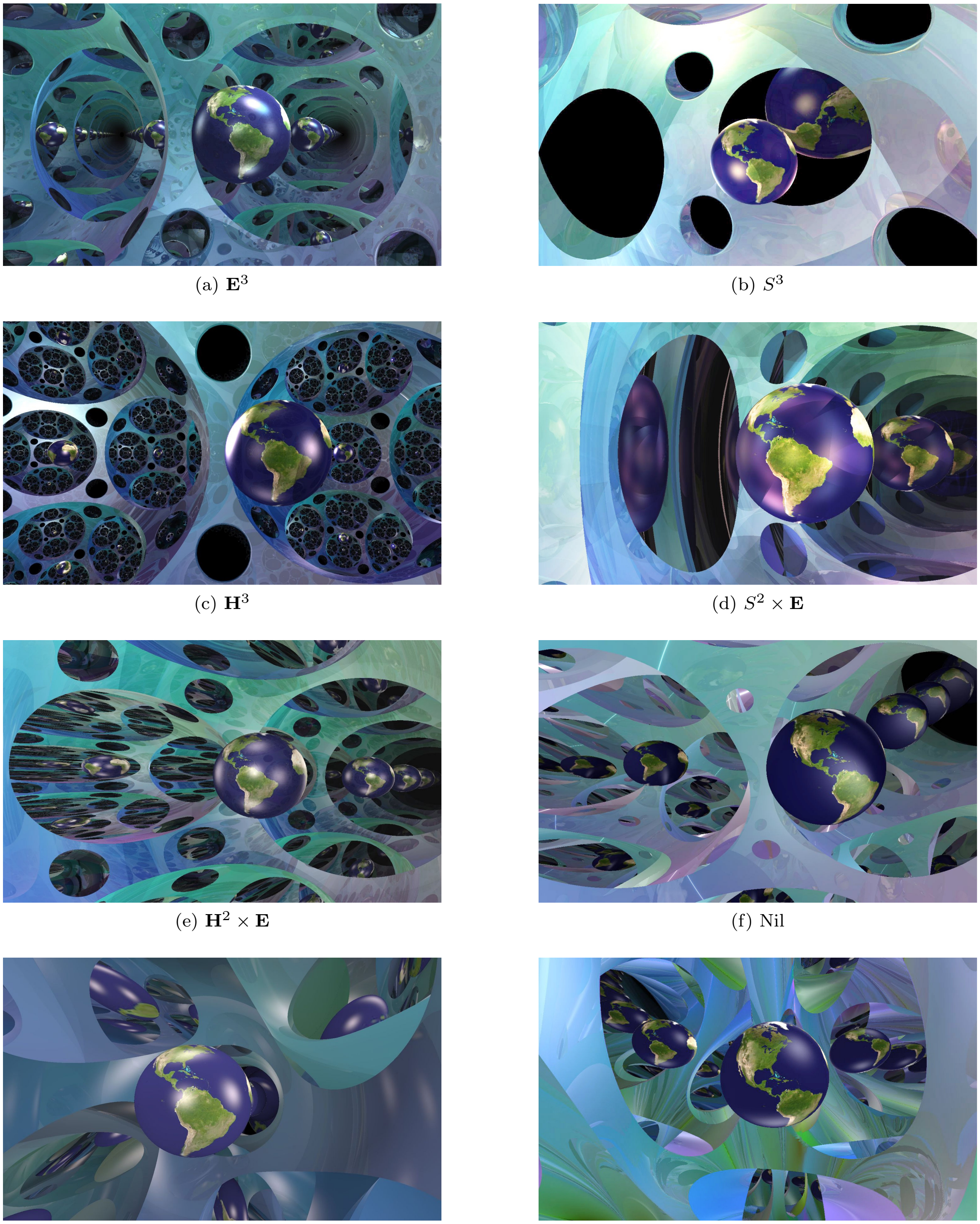
A.1 Thurston's geometries

In dimension two, Poincaré's uniformization theorem implies that every two-dimensional manifold can actually be equipped with a geometric structure modeled on one of the homogeneous spaces $\mathbf{E}^2$, S^2, or $\mathbf{H}^2$. In the 1970s and 1980s, Thurston came to realize that a similar (but more complicated) result might hold in three dimensions. His geometrization conjecture states that every prime closed three-manifold may be cut along tori into finitely many pieces, so that each of them admits a geometric structure. Geometrization was proved by Perelman in 2003 [130, 131, 132] and provides a powerful tool in three-dimensional topology. The geometries required for geometrization can be defined abstractly as follows. A *Thurston geometry* is a pair (G, X) with the following properties.

1. X is a three-dimensional connected and simply connected manifold.
2. G is a Lie group acting, transitively on X with compact point stabilizers.
3. G is not contained in any larger group of diffeomorphisms acting with compact stabilizers.
4. There is at least one compact (G, X)-manifold.

The first of these conditions rules out unnecessary duplicity in the classification: every connected (G, X)-geometry is covered by a simply connected universal covering geometry, so it suffices to consider these. The second condition is the group-theoretic way of requiring that X has a G-invariant Riemannian metric, and the third condition is just the statement that G is actually the full isometry group. A geometry satisfying (1)–(3) is called *maximal*. The fourth condition recalls our original motivation: to study geometric structures on compact manifolds in dimension three; we need only concern ourselves with geometries which can be used to build geometric structures! There are eight Thurston geometries (see Figure A.1) that can be sorted into a collection of overlapping families constructed by similar means.

1. **Isotropic geometries.** A geometry (G, X) is *isotropic* if the point stabilizer contains $O(3)$. This acts transitively on the unit tangent sphere at a point. Since directions and planes are dual to each other, any G-invariant metric on X must have constant sectional curvature. Thus, this family consists of $S^3 = (O(4), S^3)$, $\mathbf{E}^3 = (\mathbf{R}^3 \rtimes O(3), \mathbf{R}^3)$ and $\mathbf{H}^3 = (O(3,1), \mathbf{H}^3)$.



(a) $\mathbf{E}^3$

(b) S^3

(c) $\mathbf{H}^3$

(d) $S^2 \times \mathbf{E}$

(e) $\mathbf{H}^2 \times \mathbf{E}$

(f) Nil

(g) $\widetilde{\mathrm{SL}}(2, \mathbf{R})$

(h) Sol

Figure A.1 – Inside views of tilings within each of the eight Thurston geometries. Here we have chosen similar scenes to highlight the differences stemming from the geometries. Each scene is made of spheres textured as the earth and a tiling in the style of Figure A.4

2. **Products of lower dimensional geometries.** The product of any two-dimensional geometry and the unique one-dimensional geometry (denoted by $\mathbf{E}$) gives a geometry of dimension three. This family consists of the three geometries $S^2 \times \mathbf{E}, \mathbf{H}^2 \times \mathbf{E}$ and $\mathbf{E}^2 \times \mathbf{E}$. The latter is not maximal: its isometry group is contained in that of $\mathbf{E}^3$.
3. **Isometry groups of two-dimensional geometries.** Each of the two-dimensional geometries (G, X) is isotropic, so G acts transitively on the unit tangent bundle UTX. Thus we may consider the three-dimensional geometry (G, UTX), and get a maximal geometry by taking covers and extending the isometry group if necessary. This gives the geometries S^3 (built from UTS^2) and $\mathbf{E}^3$ (built from $UT\mathbf{E}^2$), as well as a new geometry: the universal cover of $\mathrm{SL}(2, \mathbf{R})$ (built from $UT\mathbf{H}^2$).
4. **Bundles over two-dimensional geometries.** Generalizing both of the previous cases, we may construct all geometries (G, X) where X has a G-invariant bundle structure over a two-dimensional geometry. This produces one new example: Nil, a line bundle over $\mathbf{E}^2$. This bundle structure has an important geometric consequence: all manifolds with these geometries are *Seifert fibered*.
5. **Three-dimensional Lie groups.** Every three-dimensional Lie group H acts on itself freely by left translation. Starting from the homogeneous geometry (H, H), we may build a maximal geometry by taking covers and extending the group of isometries, if necessary. In addition to the unit tangent bundle geometries, this construction also recovers Nil, and produces our final geometry, Sol.

For a proof that there are only eight Thurston geometries, see for example [127].

A.2 Objectives

In his work, Thurston often spoke about what it would be like to live inside of a three-manifold [154]. Following this path, we developed with Sabetta Matsumoto, Henry Segerman, and Steve Trettel a mathematical software which offers a real-time *in-space view* simulations of the eight Thurston geometries [23]. Put differently, it shows what a person living in a three-manifold would see. This tool not only works on a desktop computer, but also in combination with a virtual reality headset. When programming this software, we kept the following goals in mind.

1. Our images must be accurate – assuming that light rays travel along geodesics, there is a correct picture of what an observer inside of a given geometry would see. Our images should accurately portray this picture.
2. Real-time graphics algorithms must be very efficient in order to run at an acceptable frame rate (i.e. the number of images computed per second). This is particularly important in virtual reality – around 90 frames per second is recommended to reduce nausea. Modern graphics cards allow for the required speed, given efficient algorithms.
3. Our algorithms must allow for a full six degrees of freedom in the position and orientation of the camera, even when the simulated geometry may not have a natural corresponding isometry. Indeed, a user in a virtual reality headset can make such motions, and the view they see must react in a sensible way.
4. As much as is possible, our algorithm should be independent of the geometry being simulated. The idea here is that it should be possible to change the code in a small number of places to convert between simulations of different geometries. Compartmentalizing the code in this way will make it easier to extend it to further geometries, beyond Thurston's eight.

5. When possible, we should make our images appealing, allowing for graphical effects including lighting, shadows, reflections, fog, etc.

Some of these goals are of course in conflict. Adding features such as shadows and reflections increases the amount of work needed to be done, which can reduce the frame rate. The frame rate is also dependent on the desired screen resolution. There are many trade-offs to be made between fidelity and speed.

Remark A.2.1. Other research groups have produced images of Thurston's geometries. This project owes its existence to a long history of previous works. We can (without any attempt to be exhaustive) mention Weeks' *Curved Spaces* [155], Berger's *Éspaces imaginaires* [40] or the *HyperRogue* project [102, 103], by Kopczyński and Celińska-Kopczyńska. The novelty of our approach relies on the computer graphics techniques that we used. □

A.3 Ray-tracing vs ray-marching

In this section we explain the method we used to render the geometries. For simplicity, we only sketch the general strategy in a simply connected space, and omit the problems related to stereoscopy, user displacements, lightening, quotient manifolds/orbifolds etc.

Ray-tracing. Ray-tracing is a very common technique to render scenes in computer graphics. Following Fermat's principle, we assume that light travels along geodesics. To render an image of our scene, we place a virtual camera in the space X at a point p_0. We identify the computer screen with a portion of a hyperplane in the tangent space of X at p_0, see Figure A.2. Each pixel on the screen corresponds to a tangent vector at p_0, and so determines a geodesic ray starting at p_0. To

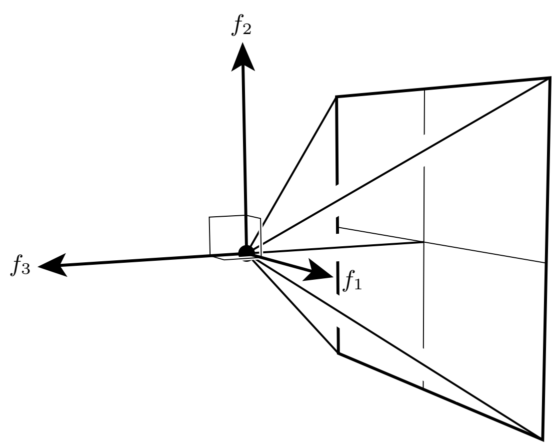


Figure A.2 – The initial tangent vector is of the form $sf_1 + tf_2 - f_3$, where s and t are coordinates on the screen.

color the pixel, we must work out what object in the scene the ray hits. In this approach objects are often approximated by a triangulation, called *mesh*. Thus it suffices to compute the (eventual) intersection of the ray with each face of each object.

In the euclidean space, this computation is very efficient as it reduces to elementary linear algebra. In practice, one does not compute the intersection with each face. There are improved algorithms that exclude a priori many triangles and thus speed up the computations. In non-euclidean geometry, this strategy is harder to implement. Indeed, the solution of the geodesic flow is sometimes rather complicated. For instance, in Sol it involves the Jacobi elliptic functions. Computing the possible intersection of the ray with all faces of all objects, even numerically, can slow down the algorithm and reduce the performances. For this reason we favored an other technique called ray-marching.

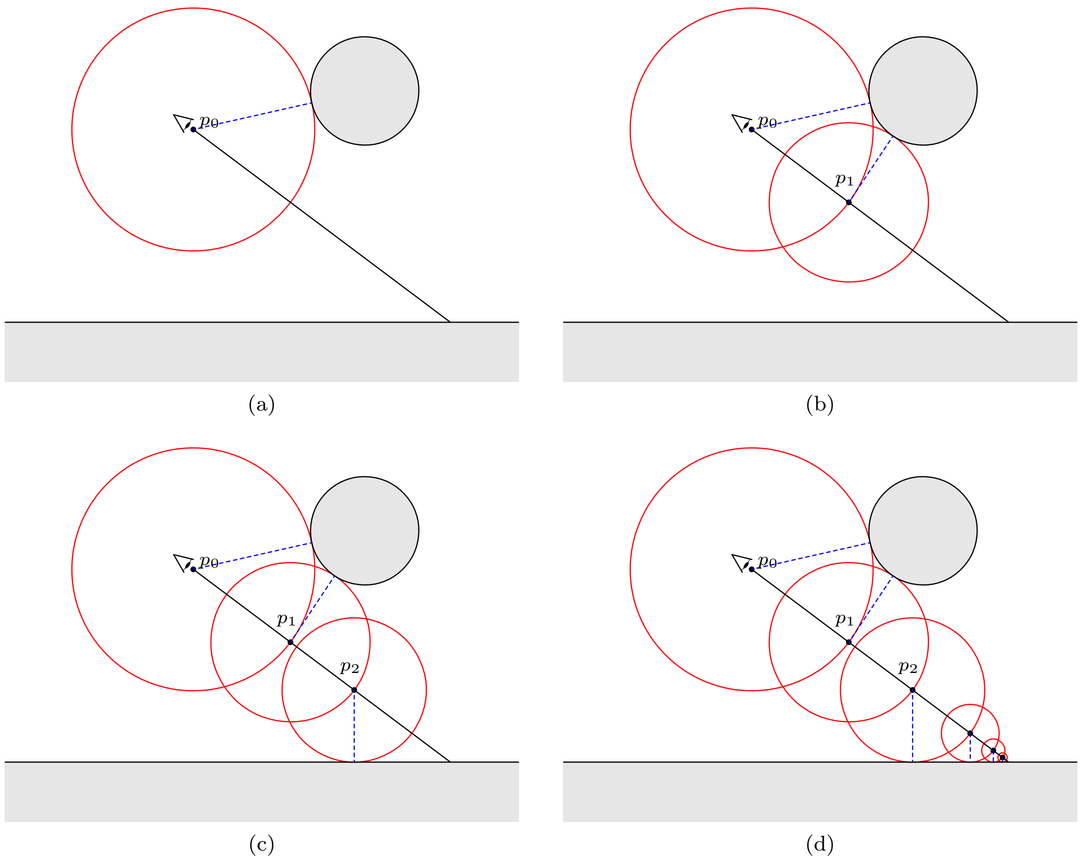


Figure A.3 – Ray-marching to find the point at which a ray hits an object, for a scene in $\mathbf{E}^2$ consisting of a disk and a half-plane.

Ray-marching. Ray-marching is a relatively new technique to produce real-time graphics using modern graphics processing units (GPU), although its roots go back to the 1980's at least [90]. Ray-marching is similar to ray-tracing in that for each pixel of the screen, we shoot a ray from a virtual camera to determine what color the pixel should be. Unlike most ray-tracing implementations however, the objects in the world are not described using meshes. Instead, one uses *signed distance functions*.

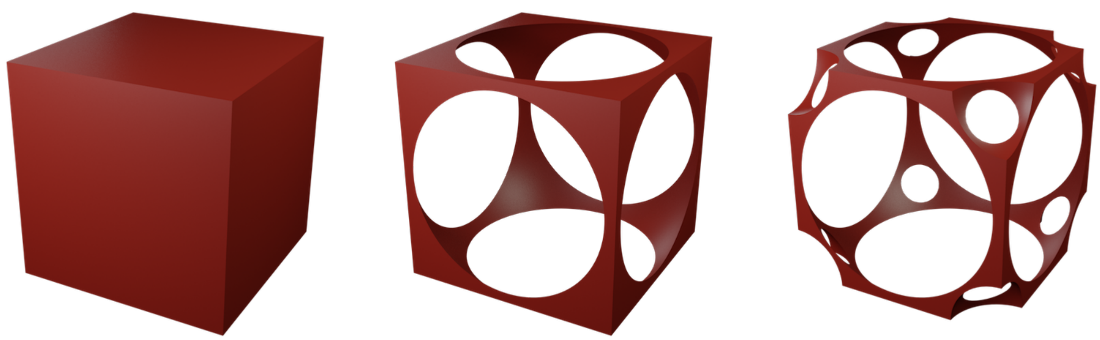
Definition A.1. Let X be the ambient space. Let S be a closed subset of X which we refer to as an *object*. The *signed distance function* of S is a map $\sigma\colon X \to \mathbf{R}$ defined as follows.

$$\sigma(p) = \begin{cases} d(p,S) & \text{if } p \in X \setminus S, \\ -d(p, X \setminus S) & \text{if } p \in S \end{cases}$$

In particular, the boundary of the object S is just the zero level set of σ. As an example, if our object S is a ball of radius r, centered at $c \in X$, then the signed distance function for S is

$$\sigma(p) = d\left(p, c\right) - r. \tag{A.1}$$

This way of representing objects is particularly adapted to Boolean operations. Consider for instance a collection of objects $(S_i)_{i\in I}$, described by signed distance functions $(\sigma_i)_{i\in I}$. The signed distance function for their union is $\min_{i\in I}\{\sigma_i\}$. Similarly the complement of an object is given by the negative of its signed distance function. For more examples of signed distance functions in $\mathbf{E}^3$, and more ways to combine them, see Quilez [134]. In our simulations, we often draw a tiling in an inexpensive manner by deleting a ball from the center of each tile. See Figure A.4.



(a) A tile.

(b) A ball is deleted from the center of the tile.

(c) A ball is deleted from the center and each vertex of the tile.

Figure A.4 – Extrinsic view of some objects with inexpensive signed distance functions for a $\mathbf{Z}^3$-invariant tiling in $\mathbf{E}^3$.

To render an image of our scene, we still identify the computer screen with a portion of a hyperplane in the tangent space of X at p_0, so that each point on the screen corresponds to a geodesic ray in the space X. The algorithm to compute the (eventual) intersection of this ray with objects is the scene is illustrated in Figure A.3. We start at p_0, the position of the camera (Figure A.3a). For simplicity, we assume that p_0 is not inside an object. We write σ for the signed distance function corresponding to the union of all objects in the scene. We evaluate σ at p_0. Since no object in the scene is within $\sigma(p_0)$ of p_0, we can safely march along our ray by a distance of $\sigma(p_0)$ without hitting anything. We call the resulting point p_1. We can then safely march forward again by $\sigma(p_1)$ to reach p_2. We repeat this procedure until either we reach a maximum number of iterations, or we reach a maximum distance, or the signed distance function evaluates to a sufficiently small threshold value, ε say. In the first two cases we color the pixel by some background color. In the third case (Figure A.3d) we declare that we have hit an object. In the latter case, we compute the color for the pixel based on which object we hit, using textures, and/or various lighting techniques, for example the Phong reflection model [133].

Distance underestimators The advantage of ray-marching is that we do not need to compute the intersection of a ray with all faces of all objects. The price to pay though is that the signed distance functions for some objects may be difficult to find or expensive to calculate. However, we do not need the exact sign distance function to run the ray-marching algorithm. This is the purpose of the next definition.

Definition A.2. Suppose that $\sigma\colon X \to \mathbf{R}$ is the signed distance function for an object S. We say that a function $\sigma'\colon X \to \mathbf{R}$ is a *distance underestimator* if

1. the signs of $\sigma'(p)$ and $\sigma(p)$ are the same for all points $p \in X$,
2. $|\sigma'(p)| \leqslant |\sigma(p)|$ for all $p \in X$, and
3. if $\{p_1, p_2, \dots\}$ is a sequence of points in X such that $\lim \sigma'(p_n) = 0$, then $\lim \sigma(p_n) = 0$.

We do not require that σ' is continuous, but the second and third conditions imply that a distance underestimator vanishes only on the boundary of S. One can prove that if we replace the signed distance function σ by a distance underestimator σ', then running the ray-marching algorithm on an ideal computer (one that is not limited by the number of steps) would produce the exact same picture. In practice, if a distance underestimator is significantly easier to compute than the signed distance function then trading an increased number of iterations for improved speed of computation can be advantageous. We extensively used this strategy in Nil, Sol or $\mathrm{SL}(2, \mathbf{R})$.

A.4 Implementation

Our software implements the ray-marching algorithm described above. This project was the occasion to take a fresh look at the Thurston geometries. Indeed, the problems we had to solve were not the ones we typically study as researchers. We faced in particular numerous numerical challenges, some of them have not been completely solved yet. For instance, in order to render accurately a scene lighted by point lights, we need to solve the following problem: given any two points $p, q \in X$, find all geodesics from p to q. This question has an easy solution in the isotropic and the product geometries. In Nil and the universal cover of $\mathrm{SL}(2, \mathbf{R})$ the situation is not so simple. Nevertheless the stabilizer of the origin contains a one parameter family of rotations. Taking advantage of this fact, we reduced the problem to finding the zeros of a one-variable convex function (which we achieved using Newton's method). Unfortunately, Sol does not have as many symmetries, and we don't have an efficient algorithm for this problem in this geometry.

The code is written in JavaScript and OpenGL so that it runs as a web application. The application is compatible with many virtual reality headsets. It offers a complete immersion into the Thurston geometries. We used this software to illustrate and explain rather strange phenomena in Nil and Sol [19, 20]. The sources, released under the terms of the GNU General Public License, are available online [21].

A.5 Outcome

We conclude this chapter with a small gallery of Thurston's geometries. More pictures and videos are available on the website of the project [22].

Euclidean space. From a group theoretic point of view, the co-compact discrete subgroups of $\mathbf{E}^3$ have been classified. These are the crystallographic groups [49]. Every finite volume euclidean three-manifold is finitely covered by the three-torus. In Figure A.5, we show the in-space view of the regular three-torus, rendered with a single ball and a multicolor collection of five lights.

The three sphere. Figure A.6 shows the lifts of some randomly chosen fibers of the unit tangent bundle over S^2 (i.e. the Hopf fibration), and their reflected images in a ball. These are the fibers of the Seifert fiber space structure on spherical three-manifolds.

Hyperbolic space. Of the eight Thurston geometries, the classification of hyperbolic manifolds (and orbifolds) is the least well understood. Figure A.7 shows the Seifert-Weber dodecahedral space, with a fundamental domain drawn in a style similar to Figure A.4b.

Product geometry: $S^2 \times \mathbf{E}$. There are only seven manifolds with the $S^2 \times \mathbf{E}$ geometry. These are listed in [144]. In Figure A.8, we show the in-space view of the Hopf manifold $S^2 \times S^1$. It shows a collection of spheres spaced at the vertices in $S^2 \times \{0\}$ of a regular dodecahedron. The observer is looking in the S^1 direction.

Product geometry: $\mathbf{H}^2 \times \mathbf{E}$. The manifolds with the $\mathbf{H}^2 \times \mathbf{E}$ geometry are classified in [144]. In Figure A.9, we show the in-space view of the orbifold $T \times S^1$, where T a torus containing a cone point of angle π. The scene is made of a slab $T \times [-\varepsilon, \varepsilon]$, with four holes cut from the fundamental domain of T, and a further hole cut around the cone point. The observer is looking in the S^1 direction.

Nil. A possible model for Nil is the Heisenberg group

$$\mathrm{Heis} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \ : \ x, y, z \in \mathbf{R} \right\}$$

We identify Nil with $\mathbf{R}^3$ via the xyz-coordinates. In Figure A.10 we represented the xy-plane textured with a grid and lightened by four color lights represented by balls. Since the observer stands far away from the lights, they appear as multiple rings. See [19] for an explanation of this phenomena. The light intensity is not a smooth function in Nil. It may blow up far away from the light source. This explains the contrast between dark and bright places on the plane.

$\widetilde{\mathrm{SL}}(2, \mathbf{R})$. In Figure A.11, we show the in-space view of a scene in $\widetilde{\mathrm{SL}}(2, \mathbf{R})$ geometry. It shows a single globe living in the unit tangent bundle for a sphere with cone points $\pi/3, \pi/3$, and $2\pi/3$.

Sol. The suspension of a two-torus by an Anosov matrix has the Sol geometry. Figure A.12 represents this manifold. The scene consists of two cubes, build as the intersection of six half-spaces.

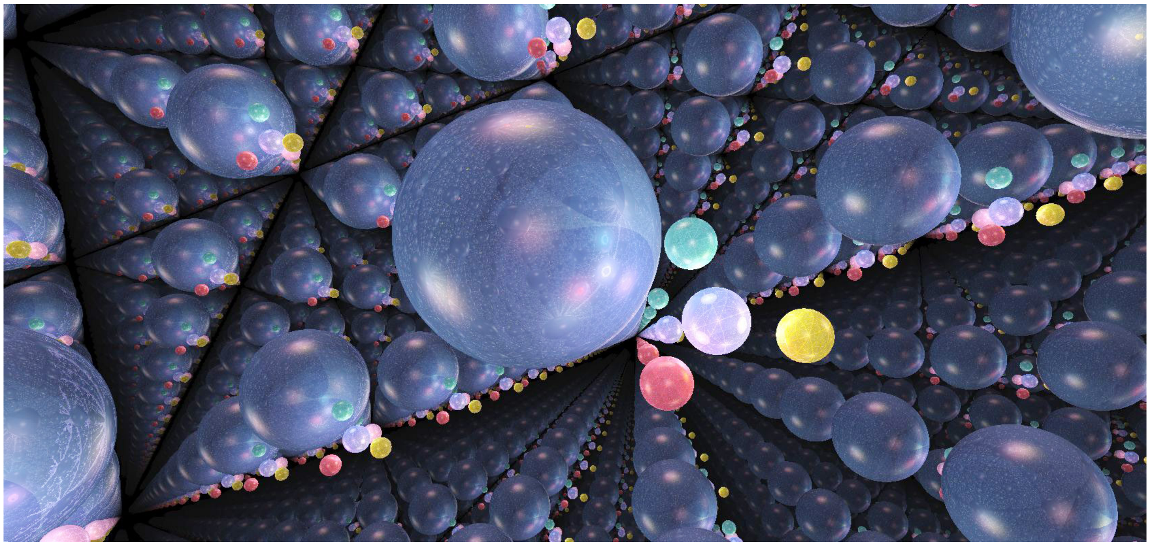


Figure A.5 – The regular three-torus, lit by a collection of lights represented by balls.

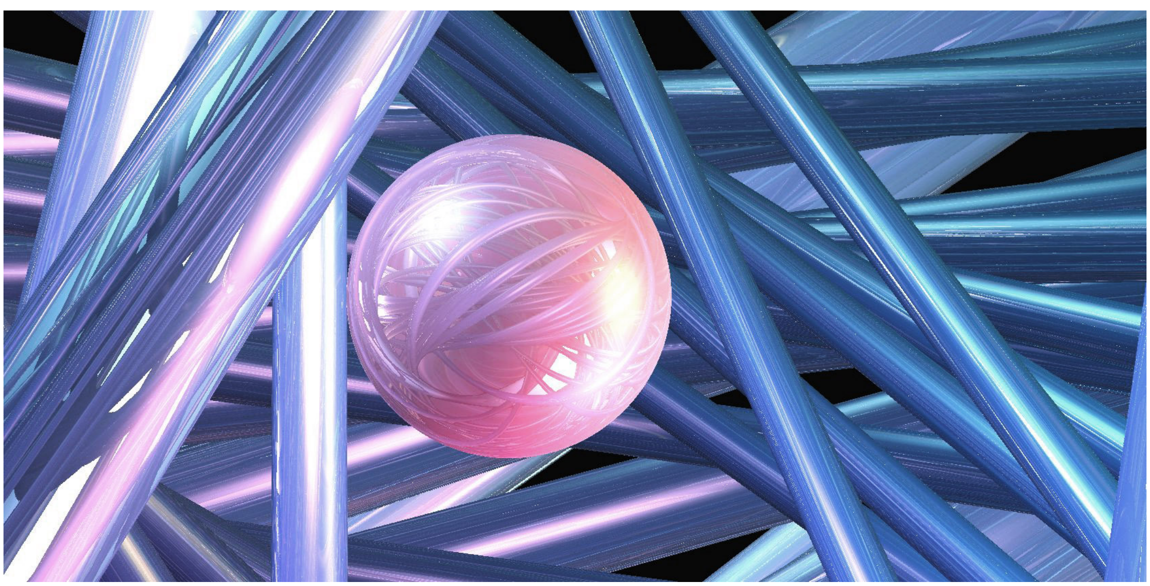


Figure A.6 – Spherical geometry (the Hopf fibration).

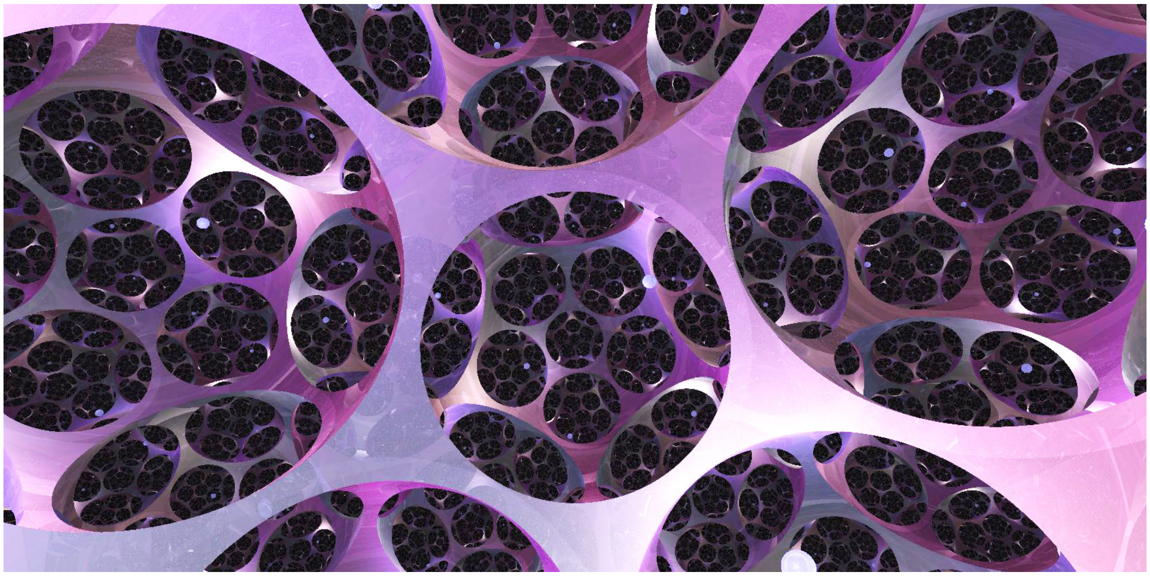


Figure A.7 – Seifert-Weber dodecahedral space.

Figure A.8 – The Hopf manifold $S^2 \times S^1$.

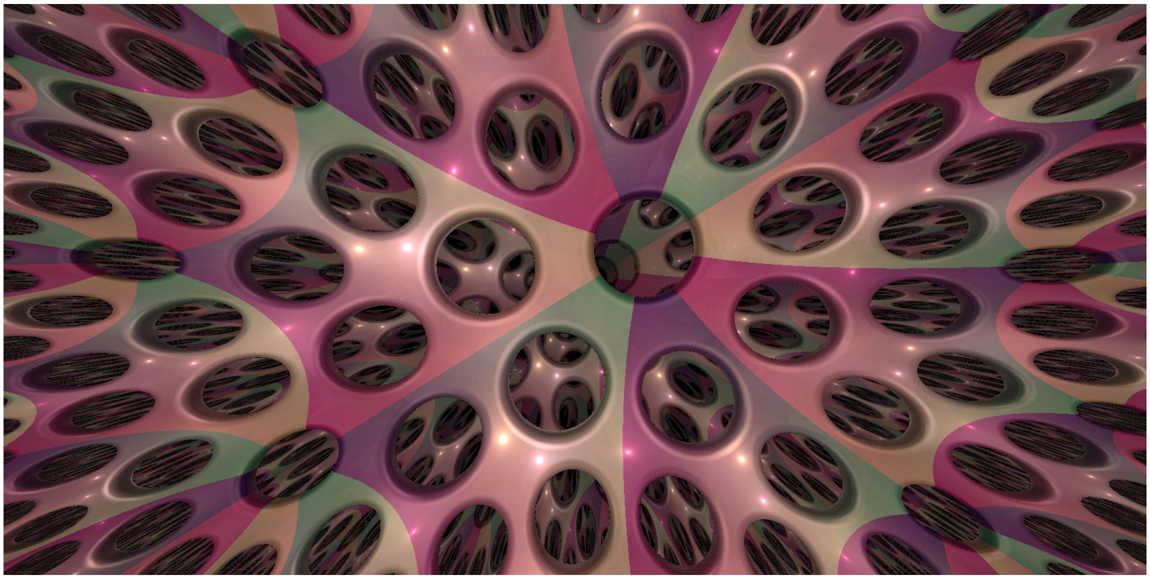


Figure A.9 – The product of a torus with cone point of angle π and a circle.

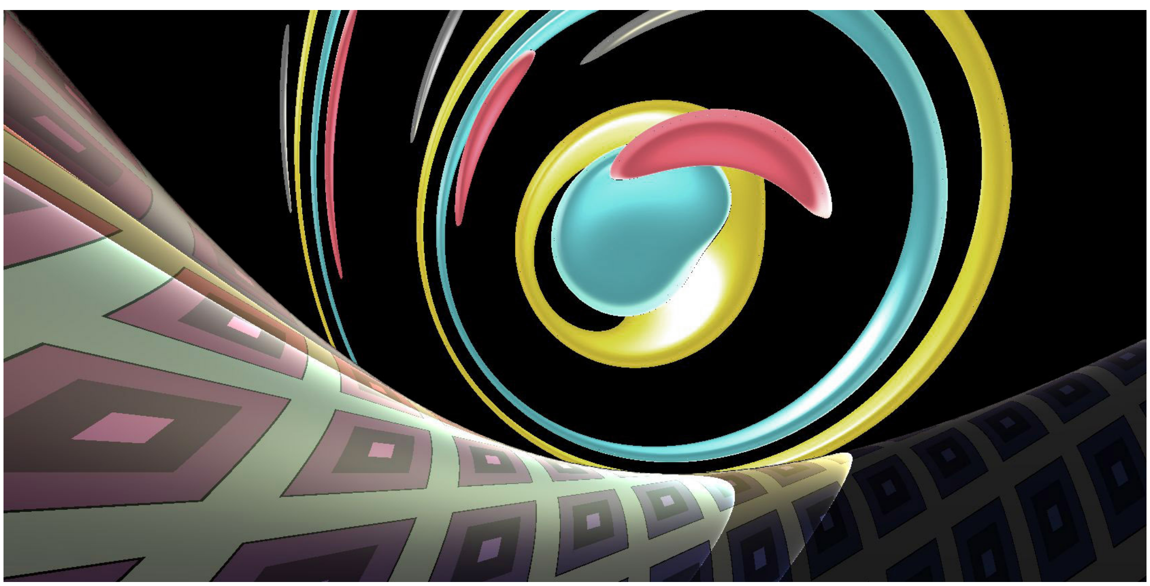


Figure A.10 – Sunset in Nil.

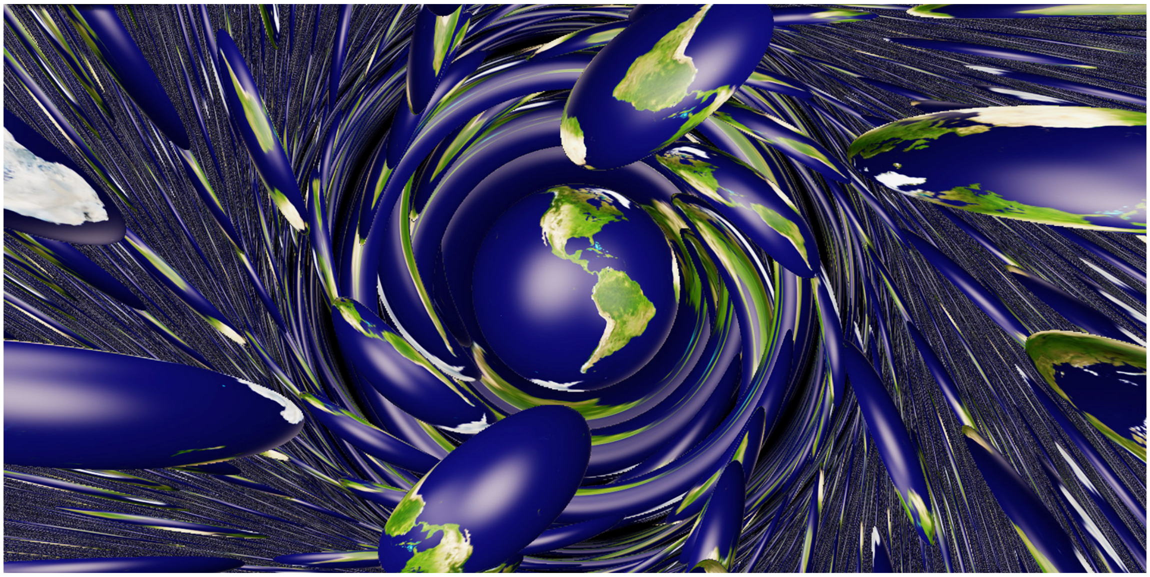


Figure A.11 – A lattice of sphere in $\widetilde{\mathrm{SL}}(2, \mathbf{R})$.

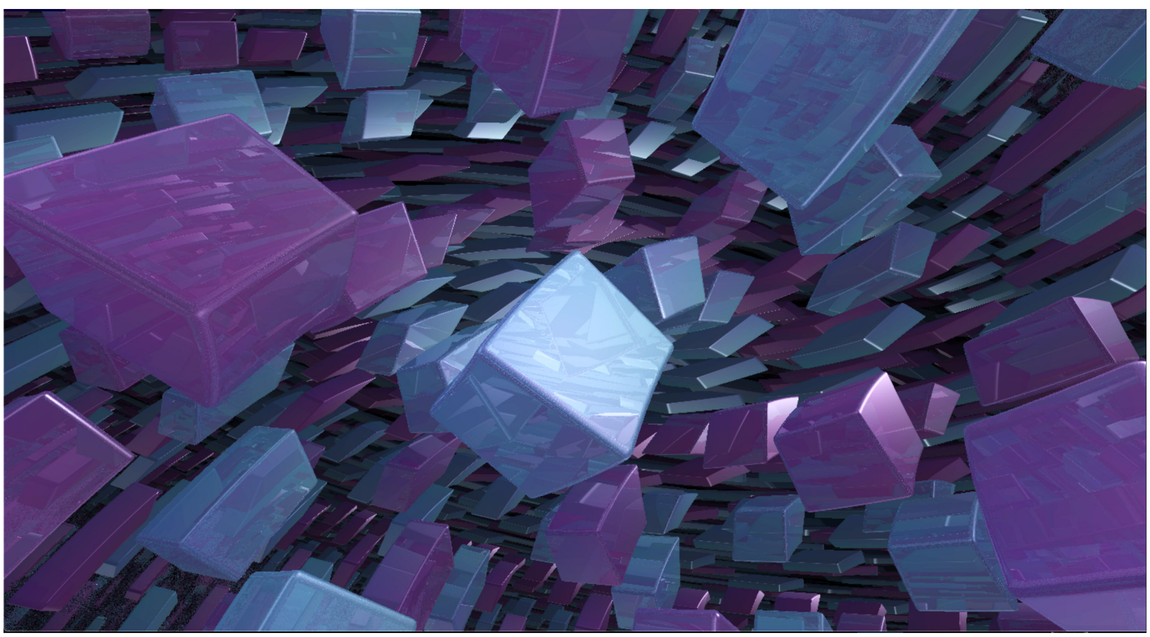


Figure A.12 – A lattice of cubes in Sol.

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